Induced and Coinduced Representations of Hopf Group Coalgebra

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Abstract

In this work we study the induction theory for Hopf group coalgebra. To reach this goal we define a substructure B of a Hopf group coalgebra H, called subHopf group coalgebra. Also, we introduced the definition of Hopf group suboalgebra and group coisotropic quantum subgroup of H.

1 Introduction

The induced representation of quantum group (quasitriangular Hopf algebra [M] and [Mon]) is introduced by Gonzalez-Ruiz, L. A. Ibort [G-I] and Ciccoli [C]. The structure of Ciccoli has many difficultites in the structure of quantum subgroups. For this, recently, Hegazi, Agauany, F. Ismail and I. Saleh in [H], have succeeded to give a new algebric structure for quantum subgroups and subquantum groups, that is a subspase B of a bialgebra H is called a sub-bi-algebra if the restriction to B of the structure of H turns B into bialgebra. But a bi-sub-algebra of H is a different thing. It is pair (B, π) consisting of another bialgebra and surjective homomorphism $\pi : H \to B$. A relation between these two notions are given if the bialgebra allows a decomposition of the form $H = B \oplus I$ with I a bi-ideal in H. The projection π into B yields a bi-sub-algebra (B, π) in the above sense. Conversely if $B \subseteq H$ is both a sub-bi-algebra and a bi-sub-algebra of H such that $\pi^2 = \pi$, then $H = B + Ker\ \pi$ and we have a decomposition of the form $H = B \oplus I$. These constractions made us able to introduce the induced representation of Hopf alagebra in each case. That is a B-comodule for a sub-Hopf algebra B of a Hopf algebra H gives rise to an induced H-Hopf module and that a

representation of a quantum subgroup (B, π) of a quantum group H induced a (so-called first type) Hopf representation of H. This procedure realizes a quantum group induced representation. These structre was given for first time by E. Tuft.

Recently, Turaev and Virelizier gives use a new definition for a generalization of Hopf algebra, for Hopf algebra structure see [S] and [Mon], i.e. Hopf group coalgebra. These generalization gives us a new quantum group structure and the algebraic structure of these structure is of great importance:

1. Hegazi and Abd Hafez introduced "multiplier Hopf group coalgebra".
2. Van Deale, Hegazi and Abd Hafez introduced the new algebraic structure "group-cograded Hopf *-algebra", J. Algebra.
3. Hegazi, studied Differential calculus on Hopf group coalgebra (M. Sc.).

In this paper, unless otherwise, every thing takes place over a field K, and K-space means vector space over K. A map f from a space V into a space W always means linear map over K. The tensor product $V \otimes W$ is understood to be $V \otimes_K W$, $I : V \to V$ always denotes the identity map, and the transposition map $\tau : V \otimes W \to W \otimes V$ is defined by $\tau(v \otimes w) = w \otimes v$ for $v \in V, w \in W$. Let $f : C \to D$ be a map. Then $f^* : D^* \to C^*$ is a map, where $f^*(\phi)(c) = \phi(f(c))$ for all $\phi \in D^*, c \in C$.

Recently, quasitriangular Hopf $\pi-$coalgebra are introduced by Truaev [T]. He gives rise to crossed π-catogeries. Virelizier [V-V1] studied the algebraic properties of the Hopf $\pi-$coalgebras, also he has show that the existence of integrals and trace for such catogery and has generlized the main properties of the quasitriangular Hopf algebra to the setting of Hopf $\pi-$coalgebra. Now, let us give some basic definitions about Hopf $\pi-$coalgebra. For group π, a π-coalgebra (over K) is a family $C = \{C_\alpha\}_{\alpha\in\pi}$ of K-spaces endowed with a family $K-$maps (the comultiplication) $\Delta = \{\Delta_{\alpha,\beta} : H_{\alpha\beta} \to H_\alpha \otimes H_\beta\}_{\alpha,\beta\epsilon\pi}$, and K-map (the counit) $\varepsilon : H_1 \to k$ such that the following diagrams are commute:

$$\begin{array}{ccc} C_{\alpha\beta\gamma} & \xrightarrow{\Delta_{\alpha,\beta\gamma}} & C_\alpha \otimes C_{\beta\gamma} \\ \Delta_{\alpha\beta,\gamma} \downarrow & & \downarrow I_\alpha \otimes \Delta_{\beta,\gamma} \\ C_{\alpha\beta} \otimes C_\gamma & \xrightarrow[\Delta_{\alpha,\beta} \otimes I_\gamma]{} & C_\alpha \otimes C_\beta \otimes C_\gamma \end{array}$$

$$(\Delta_{\alpha,\beta} \otimes I_\gamma)\Delta_{\alpha\beta,\gamma} = (I_\alpha \otimes \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma}$$

$$
\begin{tikzcd}
 & C_1 \otimes C_\alpha \arrow[dl, "\epsilon \otimes I_\alpha"'] \\
K \otimes C_\alpha & \\
 & C_\alpha \arrow[ul, "\sim_\alpha"] \arrow[uu, "\Delta_{1,\alpha}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
C_\alpha \otimes C_1 \arrow[dr, "I_\alpha \otimes \epsilon"] & \\
 & C_\alpha \otimes K \\
C_\alpha \arrow[uu, "\Delta_{\alpha,1}"] \arrow[ur, "\sim_\alpha"'] &
\end{tikzcd}
.
$$

$$(\epsilon \otimes I_\alpha)\Delta_{1,\alpha} = \sim_\alpha = (I_\alpha \otimes \epsilon)\Delta_{\alpha,1}$$

A Hopf $\pi-$coalgebra is a π-coalgebra $(H = \{H_\alpha\}_{\alpha\in\pi}, \Delta, \epsilon)$ with a family $S = \{S_\alpha : H_\alpha \to H_{\alpha^{-1}}\}_{\alpha\in\pi}$ of K-maps such that

1. H_α is an algebra with multiplication μ_α and unit $\eta_\alpha(1_K)$ for all $\alpha \in \pi$;
2. $\Delta_{\alpha,\beta}$, ϵ are algebra maps for all $\alpha, \beta \in \pi$,
3. The antipode S satisfy

$$
\begin{tikzcd}
H_1 \arrow[r, "1_\alpha\epsilon"] \arrow[d, "\Delta_{\alpha^{-1},\alpha}"'] & H_\alpha \\
H_{\alpha^{-1}} \otimes H_\alpha \arrow[r, "S_{\alpha^{-1}} \otimes I_\alpha"] & H_\alpha \otimes H_\alpha \arrow[u, "\mu_\alpha"]
\end{tikzcd}
\qquad
\begin{tikzcd}
H_1 \arrow[r, "1_\alpha\epsilon"] \arrow[d, "\Delta_{\alpha,\alpha^{-1}}"'] & H_\alpha \\
H_\alpha \otimes H_{\alpha^{-1}} \arrow[r, "I_\alpha \otimes S_{\alpha^{-1}}"] & H_\alpha \otimes H_\alpha \arrow[u, "\mu_\alpha"]
\end{tikzcd}
$$

$$\mu_\alpha(S_{\alpha^{-1}} \otimes I_\alpha)\Delta_{\alpha^{-1},\alpha} = 1_\alpha\epsilon = \mu_\alpha(I_\alpha \otimes S_{\alpha^{-1}})\Delta_{\alpha,\alpha^{-1}}$$

A Hopf π-coalgebra H is of finite type when every H_α is finite dimensional. Note that it does not means that $\oplus_{\alpha\in\pi}H_\alpha$ is finite dimensional (unless $H_\alpha = 0$ for all but a finite number of $\alpha \in \pi$). H is totally finite when the direct sum $\oplus_{\alpha\in\pi}H_\alpha$ is finite dimensional. The notion of Hopf $\pi-$coalgebra is not self dual, i.e., given a Hopf $\pi-$coalgebra $H = \{H_\alpha\}_{\alpha\in\pi}$, the family $H^* = \{H^*_\alpha\}_{\alpha\in\pi}$ does not have a natural structure of Hopf $\pi-$coalgebra. Note, $(H_1, \Delta_{1,1}, \epsilon)$ is a (classical) Hopf algebra.

A Hopf $\pi-$coalgebra H is said to have a left (right) cosection if there exist a family of algebra maps $\eta = \{\eta_\alpha : H_1 \to H_\alpha\}_{\alpha\in\pi}$ such that η_α is left (right) H_1-comodule map for all $\alpha \in \pi$ i.e., the following digram is commute

$$
\begin{tikzcd}
H_1 \arrow[r, "\eta_\alpha"] \arrow[d, "\Delta_{1,1}"'] & H_\alpha \arrow[d, "\Delta_{1,\alpha}"] \\
H_1 \otimes H_1 \arrow[r, "I_1 \otimes \eta_\alpha"'] & H_1 \otimes H_\alpha
\end{tikzcd}
$$

Let C be a π-coalgebra. A right π-comodule over C is a family $M = \{M_\alpha\}_{\alpha\in\pi}$ of K-spaces endowed with a family $\theta = \{\theta_{\alpha,\beta} : M_{\alpha\beta} \to M_\alpha \otimes C_\beta\}_{\alpha,\beta\epsilon\pi}$ of K-maps such that the following diagrams are commute:

$$\begin{array}{ccc} M_{\alpha\beta\gamma} & \xrightarrow{\theta_{\alpha,\beta\gamma}} & M_\alpha \otimes C_{\beta\gamma} \\ \theta_{\alpha\beta,\gamma}\downarrow & & \downarrow I_\alpha \otimes \Delta_{\beta,\gamma} \\ M_{\alpha\beta}\otimes C_\gamma & \xrightarrow{\theta_{\alpha,\beta}\otimes I_\gamma} & C_\alpha \otimes C_\beta \otimes C\gamma \end{array}
\qquad
\begin{array}{ccc} M_\alpha & & \\ & \searrow \theta_{\alpha,1} & \\ \sim_{M_\alpha}\downarrow & & M_\alpha \otimes C_1 \\ & \swarrow I_\alpha\otimes\epsilon & \\ M_\alpha\otimes K & & \end{array}$$

$$(\theta_{\alpha,\beta}\otimes I_\gamma\ )\theta_{\alpha\beta,\gamma} = (I_\alpha\otimes\Delta_{\beta,\gamma})\theta_{\alpha,\beta\gamma} \qquad (I_\alpha\otimes\epsilon)\theta_{\alpha,1} = \sim_{M_\alpha}$$

A right Hopf π-comodule over H is a right π-comodule $M = \{M_\alpha\}_{\alpha\in\pi}$ by coaction $\theta = \{\theta_{\alpha,\beta}\}_{\alpha,\beta\epsilon\pi}$ such that

1. M_α is $H_\alpha-$module by action ρ_α for all $\alpha \in \pi$
2. The following diagram, for all $\alpha,\beta,\gamma\in\pi$, is commute:

$$\begin{array}{ccccc} M_{\alpha\beta}\otimes H_{\alpha\beta} & \xrightarrow{\rho_{\alpha\beta}} & M_{\alpha\beta} & \xrightarrow{\theta_{\alpha,\beta}} & M_\alpha\otimes H_\beta \\ \theta_{\alpha,\beta}\otimes\Delta_{\alpha,\beta}\downarrow & & & & \uparrow \rho_\alpha\otimes\mu_\beta \\ (M_\alpha\otimes H_\beta)\otimes(H_\alpha\otimes H_\beta) & & \xrightarrow{I_\alpha\otimes\tau\otimes I_\beta} & & (M_\alpha\otimes H_\alpha)\otimes(H_\beta\otimes H_\beta) \end{array}$$

$$i.e.,\ \theta_{\alpha,\beta}\ \rho_{\alpha\beta} = (\rho_\alpha\otimes\mu_\beta)(I_\alpha\otimes\tau\otimes I_\beta)(\theta_{\alpha,\beta}\otimes\Delta_{\alpha,\beta})$$

Let C be a $\pi-$coalgebra. A $\pi-$coideal of C is a family $V = \{V_\alpha\}_{\alpha\in\pi}$ such that

1. V_α is subspace of C_α for all $\alpha\in\pi$,
2. $\Delta_{\alpha,\beta}(V_{\alpha\beta}) \subset V_\alpha\otimes C_\beta + C_\alpha\otimes V_\beta$ for all $\alpha,\beta\in\pi$.
3. $\epsilon_1(V_1) = 0$.

A Hopf π-coideal of H is a family of K-spaces $V = \{V_\alpha\}_{\alpha\in\pi}$ such that V is π-coideal of H, V_α is ideal of H_α and $S_\alpha(V_\alpha) \subseteq V_{\alpha^{-1}}$ for all $\alpha\in\pi$.

A family $A = \{A_\alpha\}_{\alpha\in\pi}$ is called subHopf $\pi-$coalgebra of H if (A,Δ,ϵ) is $\pi-$coalgebra of H, A_α is a subalgebra of H_α and $S_\alpha(A_\alpha) \subseteq A_{\alpha^{-1}}$for all $\alpha\in\pi$.

A family $A=\{A_\alpha\}_{\alpha\in\pi}$ of a subalgebra of Hopf π-coalgebra of H is called isolated if there exist a family $I=\{I_\alpha\}_{\alpha\in\pi}$ of Hopf π- coideal of H such that$H_\alpha = A_\alpha \oplus I_\alpha$for all $\alpha\in\pi$.

We will call Hopf $\pi-$subcoalgebra of H any pair (C,σ) such that $C=\{C_\alpha\}_{\alpha\in\pi}$ is a Hopf $\pi-$coalgebra and $\sigma=\{\sigma_\alpha : H_\alpha\to C_\alpha\}_{\alpha\in\pi}$ family of algebra epimorphisms which satisfies for $\alpha,\beta\in\pi$

1. $\Delta^C_{\alpha,\beta}\sigma_{\alpha\beta}=(\sigma_\alpha\otimes\sigma_\beta)\Delta^H_{\alpha,\beta}$, i.e., the following digram is commute

$$\begin{array}{ccccc} & H_{\alpha\beta} & \xrightarrow{\sigma_{\alpha,\beta}} & C_{\alpha\beta} & \\ \Delta^H_{\alpha,\beta} & \downarrow & & \downarrow & \Delta^C_{\alpha,\beta} \\ & H_\alpha\otimes H_\beta & \underrightarrow{\sigma_\alpha\otimes\sigma_\beta} & C_\alpha\otimes C_\beta & \end{array}$$

2. $\varepsilon^C\sigma_1=\varepsilon^H$, i.e., the following digram is commute

$$\begin{array}{ccccc} & H_1 & \xrightarrow{\sigma_1} & C_1 & \\ \epsilon^H & \searrow & & \swarrow & \epsilon^C \\ & & K & & \end{array}$$

3. $S^C_\alpha\sigma_\alpha=\sigma_{\alpha^{-1}}S^H_\alpha$ $(\sigma_{\beta\alpha\beta^{-1}}\varphi^H_\beta=\varphi^C_\beta\sigma_\alpha)$ i.e., the following digram is commute

$$\begin{array}{ccccc} & H_\alpha & \xrightarrow{\sigma_\alpha} & C_\alpha & \\ S^H_\alpha & \downarrow & & \downarrow & S^C_\alpha \\ & H_{\alpha^{-1}} & \underrightarrow{\sigma_{\alpha^{-1}}} & C_{\alpha^{-1}} & \end{array}$$

A pair (C,σ) is called left $\pi-$coisotropic quantum subgroup of H if

1. C is $\pi-$coalgebra,
2. C_α is left H_α-module by ω_α for all $\alpha\in\pi$,
3. $\sigma=\{\sigma_\alpha : H_\alpha\to C_\alpha\ \}_{\alpha\in\pi}$ family of surjective linear maps such that
   - (a) σ_α is left H_α-module map for all $\alpha\in\pi$,
   - (b) $\Delta^C_{\alpha,\beta}\sigma_{\alpha\beta}=(\sigma_\alpha\otimes\sigma_\beta)\Delta^H_{\alpha,\beta}$,
   - (c) $\varepsilon^C\circ\sigma_1=\varepsilon^H$.

Proposition 1.1. *Every isolated subHopf $\pi-$coalgebra is $\pi-$coisotropic quantum subgroup.*

Proof. Clear from definition. $\square$

A left $\pi-$coisotropic quantum subgroup (C,σ) of H is said to have a left section if there exist a family of linear, convolution invertible, maps $g=\{g_\alpha : C_\alpha \to H_\alpha\}_{\alpha\in\pi}$ such that

1. $g_\alpha(\sigma_\alpha(1))=1$
2. For $\alpha\in\pi, u\in H_1, c\in C_\alpha$ and $v\in\sigma_\alpha^{-1}(c)$

$$\begin{aligned}&(\sigma_1\otimes I_\alpha^H)(\mu_1\otimes I_\alpha)(I_1\otimes\tau)(\Delta_{1,\alpha}^H g_\alpha\otimes I_1)(c\otimes u)\\ =\ &(I_1\otimes g_\alpha)(\sigma_1\otimes\sigma_\alpha)(\mu_1\otimes I_\alpha)(I_1\otimes\tau)(\Delta_{1,\alpha}^H\otimes I_1)(v\otimes u)\end{aligned}$$

2 Induced representations of Hopf group coalgebra

In this section, we study the induced representation for Hopf group coalgebra. To reach this goal we use the definitions of subHopf group coalgebra, Hopf group subcoalgebra and group coisotropic quantum subgroup.

Theorem 2.1. *Let $H=\{H_\alpha\}_{\alpha\in\pi}$ be a Hopf $\pi-$coalgebra and (C,σ) be a Hopf $\pi-$subcoalgebra of H. Then (C,σ) is a left $\pi-$coisotropic quantum subgroup of H.*

Proof. We define $\omega=\{\omega_\alpha=\mu_\alpha^C(\sigma_\alpha\otimes I_\alpha): H_\alpha\otimes C_\alpha\to C_\alpha\}_{\alpha\in\pi}$ we'll prove that C_α is a left $H_\alpha-$module by ρ_α. i.e., the followig digrams are commute

$$\begin{tikzcd} H_\alpha\otimes H_\alpha\otimes C_\alpha \arrow[r, "I_\alpha\otimes\omega_\alpha"] \arrow[d, "\mu_\alpha\otimes I_\alpha"'] & H_\alpha\otimes C_\alpha \arrow[d, "\omega_\alpha"] \\ H_\alpha\otimes C_\alpha \arrow[r, "\rho_\alpha"'] & C_\alpha \end{tikzcd} \qquad \begin{tikzcd} K\otimes C_\alpha \arrow[r, "\eta_\alpha\otimes I_\alpha"] \arrow[dr, "\sim"'] & H_\alpha\otimes C_\alpha \arrow[d, "\omega_\alpha"] \\ & C_\alpha \end{tikzcd}$$

$$\begin{aligned}\omega_\alpha(I_\alpha\otimes\omega_\alpha)(h\otimes k\otimes b)&=\omega_\alpha(h\otimes\sigma_\alpha(k)b)\\&=\sigma_\alpha(h)(\sigma_\alpha(k)b)\\&=\sigma_\alpha(hk)b\\&=\omega_\alpha(hk\otimes b)\\&=\omega_\alpha(\mu_\alpha\otimes I_\alpha)(h\otimes k\otimes b)\end{aligned}$$

and

$$\begin{aligned}
\omega_\alpha(\eta_\alpha \otimes I_\alpha)(k \otimes b) &= \omega_\alpha(\eta_\alpha(k) \otimes b) \\
&= \sigma_\alpha(\eta_\alpha(k))b \\
&= kb \\
&= \sim (k \otimes b)
\end{aligned}$$

Now, we'll prove that σ_α is left H_α-module map for all $\alpha \in \pi$ i.e., the following digram is commute

$$\begin{array}{cccc}
 & H_\alpha & \overset{\sigma_\alpha}{\rightarrow} & C_\alpha \\
 & & & \\
\mu^H_\alpha & \uparrow & & \uparrow \omega_\alpha \\
 & H_\alpha \otimes H_\alpha & \underset{I_\alpha \otimes \sigma_\alpha}{\longrightarrow} & H_\alpha \otimes C_\alpha
\end{array}$$

$$\omega_\alpha(I_\alpha \otimes \sigma_\alpha) = \mu^C_\alpha(\sigma_\alpha \otimes I_\alpha)(I_\alpha \otimes \sigma_\alpha) = \mu^C_\alpha(\sigma_\alpha \otimes \sigma_\alpha) = \sigma_\alpha \mu^H_\alpha.$$

□

Remark 2.2. If (C, σ) is a Hopf $\pi-$subcoalgebra of H, then the map

$$L_{\alpha,\beta} = (\sigma_\alpha \otimes I^H_\beta)\Delta^H_{\alpha,\beta} : H_{\alpha\beta} \to C_\alpha \otimes H_\beta \ \ \forall \alpha, \beta \in \pi$$

is an algebra map.

Proposition 2.3. *Let $H = \{H_\alpha\}_{\alpha\in\pi}$ be a Hopf $\pi-$coalgebra and (C, σ) be a Hopf $\pi-$subcoalgebra of H. Let $B = \{B_\alpha\}_{\alpha\in\pi}$, where $B_\alpha = \{h \in H_\alpha : L_{1,\alpha}(h) = 1 \otimes h\}$, then B_α is subalgebra of H_α and B is right $\pi-$coideal of H (this B is called $\pi-$quantum right embeddable homogeneous space of H).*

Proof.

1. Let $h, k \in B_\alpha$, then

$$L_{1,\alpha}(hk) = L_{1,\alpha}(h) \cdot L_{1,\alpha}(k) = (1 \otimes h) \cdot (1 \otimes k) = 1 \otimes hk$$

and hence $hk \in B_\alpha$.

2. Let $h \in B_{\alpha\beta}$, we‘ll prove that $\Delta^H_{\alpha,\beta}(h) \in B_\alpha \otimes H_\beta$.

$$\begin{aligned}
(L_{1,\alpha} \otimes I^H_\beta)\Delta^H_{\alpha,\beta}(h) &= ((\sigma_1 \otimes I^H_\alpha)\Delta^H_{1,\alpha} \otimes I^H_\beta)\Delta^H_{\alpha,\beta}(h) \\
&= (\sigma_1 \otimes I^H_\alpha \otimes I^H_\beta)(\Delta^H_{1,\alpha} \otimes I^H_\beta)\Delta^H_{\alpha,\beta}(h) \\
&= (\sigma_1 \otimes I^H_\alpha \otimes I^H_\beta)(I^H_1 \otimes \Delta^H_{\alpha,\beta})\Delta^H_{1,\alpha\beta}(h) \\
&= (I^C_1 \otimes \Delta^H_{\alpha,\beta})(\sigma_1 \otimes I^H_{\alpha\beta})\Delta^H_{1,\alpha\beta}(h) \\
&= (I^C_1 \otimes \Delta^H_{\alpha,\beta})(1 \otimes h) \\
&= 1 \otimes \Delta^H_{1,\alpha\beta}(h)
\end{aligned}$$

□

Lemma 2.4. *Let $H = \{H_\alpha\}_{\alpha\in\pi}$ be a Hopf $\pi-$coalgebra and let (C, σ) be a left $\pi-$coisotropic quantum subgroup of H. For $\alpha, \beta, \gamma \in \pi$ and $a, b \in H_{\alpha\beta}$ we have*

- $L_{\alpha,\beta}(ab) = \Delta^H_{\alpha,\beta}(a)\Theta L_{\alpha,\beta}(b)$
- $(I_\alpha \otimes \Delta^H_{\beta,\gamma})L_{\alpha,\beta\gamma} = (L_{\alpha,\beta} \otimes I_\gamma)\Delta^H_{\alpha\beta,\gamma}$,

where $(m \otimes n)\Theta(u \otimes v) = \omega_\alpha(m \otimes u) \otimes nv, m, u \in H_\alpha, n, v \in H_\beta$.

Proof.

$$\begin{aligned}
L_{\alpha,\beta}(ab) &= (\sigma_\alpha \otimes I^H_\beta)\Delta^H_{\alpha,\beta}\mu_{\alpha\beta}(a \otimes b) \\
&= (\sigma_\alpha \otimes I^H_\beta)(\mu_\alpha \otimes \mu_\beta)(I \otimes \tau \otimes I)(\Delta^H_{\alpha,\beta} \otimes \Delta^H_{\alpha,\beta})(a \otimes b) \\
&= (\sigma_\alpha\mu_\alpha \otimes \mu_\beta)(I \otimes \tau \otimes I)(\Delta^H_{\alpha,\beta} \otimes \Delta^H_{\alpha,\beta})(a \otimes b) \\
&= (\sigma_\alpha\mu_\alpha \otimes \mu_\beta)(I \otimes \tau \otimes I)(\Delta^H_{\alpha,\beta} \otimes \Delta^H_{\alpha,\beta})(a \otimes b) \\
&= (\omega_\alpha(I_\alpha \otimes \sigma_\alpha) \otimes \mu_\beta)(a^\alpha_1 \otimes a^\beta_2 \otimes b^\alpha_1 \otimes b^\beta_2) \\
&= \omega_\alpha(a^\alpha_1 \otimes \sigma_\alpha(b^\alpha_1)) \otimes a^\beta_2 b^\beta_2 \\
&= \Delta^H_{\alpha,\beta}(a)\Theta L_{\alpha,\beta}(b)
\end{aligned}$$

Also,

$$\begin{aligned}
(I_\alpha \otimes \Delta^H_{\beta,\gamma})L_{\alpha,\beta\gamma} &= (I_\alpha \otimes \Delta^H_{\beta,\gamma})(\sigma_\alpha \otimes I^H_{\beta\gamma})\Delta^H_{\alpha,\beta\gamma} \\
&= (\sigma_\alpha \otimes I^H_\beta \otimes I^H_\gamma)(I_\alpha \otimes \Delta^H_{\beta,\gamma})\Delta^H_{\alpha,\beta\gamma} \\
&= (\sigma_\alpha \otimes I^H_\beta \otimes I^H_\gamma)(\Delta^H_{\alpha,\beta} \otimes I_\gamma)\Delta^H_{\alpha\beta,\gamma} \\
&= ((\sigma_\alpha \otimes I^H_\beta)\Delta^H_{\alpha,\beta} \otimes I^H_\gamma)\Delta^H_{\alpha\beta,\gamma} \\
&= (L_{\alpha,\beta} \otimes I_\gamma)\Delta^H_{\alpha\beta,\gamma}
\end{aligned}$$

□

Proposition 2.5. *Let $H = \{H_\alpha\}_{\alpha\in\pi}$ be a Hopf $\pi-$coalgebra and let (C,σ) be a left $\pi-$coisotropic quantum subgroup of H. Let $G = \{G_\alpha\}_{\alpha\in\pi}$, where $G_\alpha = \{h \in H_\alpha : L_{1,\alpha}(h) = \sigma_1(1) \otimes h\}$, then G_α is subalgebra of H_α and G is right $\pi-$ coideal of H.*

Proof. For $h, g \in G_\alpha$, we have

1. $L_{1,\alpha}(h) = \sigma_1(1) \otimes h$ and $L_{1,\alpha}(g) = \sigma_1(1) \otimes g$
2. $\omega_1(I_1 \otimes \sigma_1) = \sigma_1\mu_1$

We'll prove that $hg \in G_\alpha, i.e., L_{1,\alpha}(hg) = \sigma_1(1) \otimes hg$

$$\begin{aligned}
L_{1,\alpha}(hg) &= \Delta_{1,\alpha}(h)\Theta L_{1,\alpha}(g) &&\text{by Lemma 2.4}\\
&= \Delta_{1,\alpha}(h)\Theta(\sigma_1(1) \otimes g)\\
&= (\omega_1 \otimes \mu_\alpha)(h_1^1 \otimes \sigma_1(1) \otimes h_2^\alpha \otimes g)\\
&= \omega_1(I_1 \otimes \sigma_1)(h_1^1 \otimes 1) \otimes h_2^\alpha g\\
&= \sigma_1\mu_1(h_1^1 \otimes 1) \otimes h_2^\alpha g\\
&= \sigma_1(h_1^1) \otimes h_2^\alpha g\\
&= (\sigma_1 \otimes \mu_\alpha)(\Delta_{1,\alpha}^H \otimes I_\alpha)(h \otimes g)\\
&= (I_1 \otimes \mu_\alpha)((\sigma_1 \otimes I_1)\Delta_{1,\alpha}^H(h) \otimes g)\\
&= (I_1 \otimes \mu_\alpha)(\sigma_1(1) \otimes h \otimes g)\\
&= \sigma_1(1) \otimes hg
\end{aligned}$$

Now, we'll prove that G is right π-coideal of H, i.e., for $h \in G_{\alpha\beta}, \Delta_{\alpha,\beta}^H(h) \in G_\alpha \otimes H_\beta$

$$\begin{aligned}
(L_{1,\alpha} \otimes I_\beta^H)\Delta_{\alpha,\beta}^H(h) &= ((\sigma_1 \otimes I_\alpha^H)\Delta_{1,\alpha}^H \otimes I_\beta^H)\Delta_{\alpha,\beta}^H(h)\\
&= (\sigma_1 \otimes I_\alpha^H \otimes I_\beta^H)(\Delta_{1,\alpha}^H \otimes I_\beta^H)\Delta_{\alpha,\beta}^H(h)\\
&= (\sigma_1 \otimes I_\alpha^H \otimes I_\beta^H)(I_1^H \otimes \Delta_{\alpha,\beta}^H)\Delta_{1,\alpha\beta}^H(h)\\
&= (I_1^C \otimes \Delta_{\alpha,\beta}^H)(\sigma_1 \otimes I_{\alpha\beta}^H)\Delta_{1,\alpha\beta}^H(h)\\
&= (I_1^C \otimes \Delta_{\alpha,\beta}^H)(\sigma_1(1) \otimes h)\\
&= \sigma_1(1) \otimes \Delta_{1,\alpha\beta}^H(h)
\end{aligned}$$

□

Theorem 2.6. *Suppose $V = \{V_\alpha\}_{\alpha\in\pi}$, be a right $\pi-$comodule over the left $\pi-$coisotropic quantum subgroup C of Hopf $\pi-$coalgebra H by $\rho = \{\rho_{\alpha,\beta} : V_{\alpha\beta} \to V_\alpha \otimes C_\beta\}_{\alpha,\beta\in\pi}$. Then $Ind(\rho) = \{Ind(\rho)_\alpha\}_{\alpha\in\pi}$ where $Ind(\rho)_\alpha = \{x \in V_1 \otimes H_\alpha : (I_1 \otimes L_{1,\alpha})x = (\rho_{1,1} \otimes I_\alpha)x\}$ is right $\pi-$comodule over H by $(I \otimes \Delta) = \{(I \otimes \Delta)_{\alpha,\beta} = I_1 \otimes \Delta_{\alpha,\beta}\}_{\alpha,\beta\in\pi}$*

Proof. We will prove that, for $\alpha, \beta \in \pi$,

$$(I \otimes \Delta)_{\alpha,\beta}(Ind(\rho)_{\alpha\beta}) \subseteq Ind(\rho)_\alpha \otimes H_\beta.$$

Since, for $v \otimes h \in Ind(\rho)_{\alpha\beta}$, we have

$$\begin{aligned}
(I_1 \otimes L_{1,\alpha} \otimes I_\beta)(I_1 \otimes \Delta_{\alpha,\beta})(v \otimes h) &= (I_1 \otimes (L_{1,\alpha} \otimes I_\beta)\Delta_{\alpha,\beta})(v \otimes h) \\
&= (I_1 \otimes (I_1 \otimes \Delta_{\alpha,\beta})L_{1,\alpha\beta})(v \otimes h) \qquad \text{by Lemma 2.4} \\
&= (I_1 \otimes I_1 \otimes \Delta_{\alpha,\beta})(I_1 \otimes L_{1,\alpha\beta})(v \otimes h) \\
&= (I_1 \otimes I_1 \otimes \Delta_{\alpha,\beta})(\rho_{1,1} \otimes I_{\alpha\beta})(v \otimes h) \\
&= (\rho_{1,1} \otimes I_\alpha \otimes I_\beta)(I_1 \otimes \Delta_{\alpha,\beta})(v \otimes h)
\end{aligned}$$

Now, we'll prove that the following digrams are commute

$$\begin{array}{ccc}
Ind(\rho)_{\alpha\beta\gamma} & \overset{(I\otimes\Delta)_{\alpha\beta,\gamma}}{\longrightarrow} & Ind(\rho)_{\alpha\beta} \otimes H_\gamma \\
(I \otimes \Delta)_{\alpha,\beta\gamma} \downarrow & & \downarrow (I \otimes \Delta)_{\alpha,\beta} \otimes I_\gamma \\
Ind(\rho)_\alpha \otimes H_{\beta\gamma} & \underset{I_{Ind(\rho)_\alpha} \otimes \Delta_{\beta,\gamma}}{\longrightarrow} & Ind(\rho)_\alpha \otimes H_\beta \otimes H_\gamma
\end{array}$$

and

$$\begin{array}{ccc}
Ind(\rho)_\alpha & \overset{(I \otimes \Delta)_{\alpha,1}}{\rightarrow} & Ind(\rho)_\alpha \otimes H_1 \\
\sim_{Ind(\rho)_\alpha} \searrow & & \swarrow I_{Ind(\rho)_\alpha} \otimes \epsilon^H \\
& Ind(\rho)_\alpha \otimes K &
\end{array}$$

$$\begin{aligned}
((I \otimes \Delta)_{\alpha,\beta} \otimes I_\gamma)(I \otimes \Delta)_{\alpha\beta,\gamma} &= (I_1 \otimes \Delta_{\alpha,\beta} \otimes I_\gamma)(I_1 \otimes \Delta_{\alpha\beta,\gamma}) \\
&= (I_1 \otimes (\Delta_{\alpha,\beta} \otimes I_\gamma)\Delta_{\alpha\beta,\gamma}) \\
&= (I_1 \otimes (I_\alpha \otimes \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma}) \\
&= (I_1 \otimes I_\alpha \otimes \Delta_{\beta,\gamma})(I_1 \otimes \Delta_{\alpha,\beta\gamma}) \\
&= (I_{Ind(\rho)_\alpha} \otimes \Delta_{\beta,\gamma})(I \otimes \Delta)_{\alpha,\beta\gamma}
\end{aligned}$$

$$\begin{aligned}
(I_{Ind(\rho)_\alpha} \otimes \epsilon^H)(I \otimes \Delta)_{\alpha,1} &= (I_1 \otimes I_\alpha \otimes \epsilon^H)(I_1 \otimes \Delta_{\alpha,1}) \\
&= (I_1 \otimes (I_\alpha \otimes \epsilon)\Delta_{\alpha,1}) \\
&= (I_1 \otimes \sim_{H_\alpha}) \;\; = \sim_{Ind(\rho)_\alpha}
\end{aligned}$$

$\square$

Remark 2.7. Given a right corepresentation ρ of the $\pi-$coisotropic quantum subgroup of (C, σ) the corresponding corepresentation $(I \otimes \Delta)$ on $Ind(\rho)$ of H is called induced representation from ρ on H.

3 Geometric realization for induced representation

Throughout this section H is Hopf $\pi-$ coalgebra, (C, σ) is Hopf $\pi-$subcoalgebra of H and $V = \{V_\alpha\}_{\alpha\in\pi}$ be a right $\pi-$comodule over C. The purpose of this section is to explicit that the induced representation $Ind(\rho)$ from Hopf group subcoalgebra H is isomorphic to the tensor product of $\pi-$quantum embeddable homogeneous space B (in Proposition 2.3) with the given comodule V as module and in case (C, σ) is left $\pi-$coisotropic quantum subgroup H is isomorphic to $C \otimes G$ as vector space where $G = \{G_\alpha\}_{\alpha\in\pi}$, where

$$G_\alpha = \{h \in H_\alpha : L_{1,\alpha}(h) = (\sigma_1 \otimes I^H_\alpha)\Delta^H_{1,\alpha}(h) = \sigma_1(1) \otimes h\}.$$

Lemma 3.1. *$Ind(\rho)_\alpha$ is a right B_α-module for all $\alpha \in \pi$.*

Proof. Let $v_1 \otimes h_\alpha \in Ind(\rho)_\alpha, b_\alpha \in B_\alpha$, we define the right action as follows

$$\lambda_\alpha(v_1 \otimes h_\alpha \otimes b_\alpha) = v_1 \otimes h_\alpha b_\alpha$$

We need only to prove that

$$\lambda_\alpha(v_1 \otimes h_\alpha \otimes b_\alpha) = v_1 \otimes b_\alpha h_\alpha \in Ind(\rho)_\alpha.$$

We have

$$(I_1 \otimes L_{1,\alpha})(v_1 \otimes h_\alpha) = (\rho_{1,1} \otimes I_\alpha)(v_1 \otimes h_\alpha)$$

imply that

$$((I_1 \otimes L_{1,\alpha})(v_1 \otimes h_\alpha))(1 \otimes 1 \otimes b_\alpha) = ((\rho_{1,1} \otimes I_\alpha)(v_1 \otimes h_\alpha))(1 \otimes 1 \otimes b_\alpha)$$

imply that

$$v_1 \otimes \sigma_1((h_\alpha)_1) \otimes (h_\alpha)_2 b_\alpha = \rho_{1,1}(v_1) \otimes h_\alpha b_\alpha. \tag{3.1}$$

Also, we have

$$\begin{aligned} L_{1,\alpha}(h_\alpha b_\alpha) &= \Delta^H_{1,\alpha}(h_\alpha)\Theta L_{1,\alpha}(b_\alpha) \quad \text{by Lemma 2.5} \\ &= ((h_\alpha)_1 \otimes (h_\alpha)_2)\Theta(1_{C_1} \otimes b_\alpha) \\ &= \omega_\alpha((h_\alpha)_1 \otimes \sigma_1(1_{H_1})) \otimes (h_\alpha)_2 b_\alpha \\ &= \sigma_1(h_{\alpha_1}) \otimes (h_\alpha)_2 b_\alpha. \end{aligned} \tag{3.2}$$

Now,

$$\begin{aligned}(I_1 \otimes L_{1,\alpha})\lambda_\alpha(v_1 \otimes h_\alpha \otimes b_\alpha) &= (I_1 \otimes L_{1,\alpha})(v_1 \otimes h_\alpha b_\alpha) \\ &= v_1 \otimes L_{1,\alpha}(h_\alpha b_\alpha) \\ &= v_1 \otimes \sigma_1((h_\alpha)_1) \otimes (h_\alpha)_2 b_\alpha && \text{by Equation 3.2} \\ &= \rho_{1,1}(v_1) \otimes h_\alpha b_\alpha && \text{by Equation 3.1} \\ &= (\rho_{1,1} \otimes I_\alpha)(v_1 \otimes h_\alpha b_\alpha) \\ &= (\rho_{1,1} \otimes I_\alpha)\lambda_\alpha(v_1 \otimes h_\alpha \otimes b_\alpha).\end{aligned}$$

The left action is similar. □

Definition 3.2. Let (C, σ) be a Hopf $\pi-$subcoalgebra of H. If $g = \{g_\alpha : C_\alpha \to H_\alpha\}_{\alpha\in\pi}$, is a family of linear maps, its convolution inverse, if it exists, is a family of linear maps $g^{-1} = \{g_\alpha^{-1} : C_{\alpha^{-1}} \to H_\alpha\}_{\alpha\in\pi}$ such that

$$\mu_\alpha(g_\alpha \otimes g_\alpha^{-1})\Delta^C_{\alpha,\alpha^{-1}}(c) = \epsilon^C(c)1_{H_\alpha} = \mu_\alpha(g_\alpha^{-1} \otimes g_\alpha)\Delta^C_{\alpha^{-1},\alpha}(c)$$

Definition 3.3. A Hopf $\pi-$subcoalgebra (C, σ) of H is said to have a left section if there exists a family of linear, convolution invertible, maps $g = \{g_\alpha : C_\alpha \to H_\alpha\}_{\alpha\in\pi}$ such that for all $\alpha \in \pi$,

1. $g_\alpha(1) = 1$
2. $L_{1,\alpha}g_\alpha = (I_1 \otimes g_\alpha)\Delta^C_{1,\alpha}$

Lemma 3.4. *For any Hopf $\pi-$coalgebra H we have*

1. $(\Delta^H_{1,\alpha} \otimes \Delta^H_{\alpha^{-1},1})\Delta^H_{\alpha,\alpha^{-1}} = (I_1 \otimes \Delta^H_{\alpha,\alpha^{-1}} \otimes I_1)(\Delta^H_{1,1} \otimes I_1)\Delta^H_{1,1}$
2. $(\Delta^H_{\alpha^{-1},1} \otimes \Delta^H_{1,\alpha})\Delta^H_{\alpha^{-1},\alpha} = (I_{\alpha^{-1}} \otimes \Delta^H_{1,1} \otimes I_\alpha)(\Delta^H_{\alpha^{-1},1} \otimes I_\alpha)\Delta^H_{\alpha^{-1},\alpha}$

Proof.

$$\begin{aligned}(\Delta^H_{1,\alpha} \otimes \Delta^H_{\alpha^{-1},1})\Delta^H_{\alpha,\alpha^{-1}} &= (\Delta^H_{1,\alpha} \otimes I_{\alpha^{-1}} \otimes I_1)(I_\alpha \otimes \Delta^H_{\alpha^{-1},1})\Delta^H_{\alpha,\alpha^{-1}} \\ &= (\Delta^H_{1,\alpha} \otimes I_{\alpha^{-1}} \otimes I_1)(\Delta^H_{\alpha,\alpha^{-1}} \otimes I_1)\Delta^H_{1,1} \\ &= ((\Delta^H_{1,\alpha} \otimes I_{\alpha^{-1}})\Delta^H_{\alpha,\alpha^{-1}} \otimes I_1)\Delta^H_{1,1} \\ &= ((I_1 \otimes \Delta^H_{\alpha,\alpha^{-1}})\Delta^H_{1,1} \otimes I_1)\Delta^H_{1,1} \\ &= (I_1 \otimes \Delta^H_{\alpha,\alpha^{-1}} \otimes I_1)(\Delta^H_{1,1} \otimes I_1)\Delta^H_{1,1}\end{aligned}$$

$$i.e., (\Delta^H_{1,\alpha} \otimes \Delta^H_{\alpha^{-1},1})\Delta^H_{\alpha,\alpha^{-1}}(h) = h^1_{11} \otimes h^\alpha_{121} \otimes h^{\alpha^{-1}}_{122} \otimes h^1_2$$

The second statement is similar. □

Now, from Lemma 3.5 up to Theorem 3.12 below (C, σ) have a section $g = \{g_\alpha\}_{\alpha\in\pi}$ and antipode $S^C = \{S^C_\alpha : C_\alpha \to C_{\alpha^{-1}}\}_{\alpha\in\pi}$

Lemma 3.5. *For* $\alpha \in \pi, L_{1,\alpha}g_\alpha^{-1} = (S^C_1 \otimes g_\alpha^{-1})\tau\Delta^C_{\alpha^{-1},1}$

Proof. We'll prove that for $\alpha \in \pi$, $L_{1,\alpha}g_\alpha^{-1}$ and $(S^C_1 \otimes g_\alpha^{-1})\tau\Delta^C_{\alpha^{-1},1}$ are inverse to the same element $L_{1,\alpha}g_\alpha$ in the convolution algebra $Conv(C, C_1 \otimes H_\alpha)$.

$$\begin{aligned}
(L_{1,\alpha}g_\alpha * L_{1,\alpha}g_\alpha^{-1})(c) &= \{(\sigma_1 \otimes I^H_\alpha)\Delta^H_{1,\alpha}g_\alpha * (\sigma_1 \otimes I^H_\alpha)\Delta^H_{1,\alpha}g_\alpha^{-1}\}(c)\\
&= \mu_{C_1\otimes H_\alpha}\{(\sigma_1 \otimes I^H_\alpha)\Delta^H_{1,\alpha}g_\alpha \otimes (\sigma_1 \otimes I^H_\alpha)\Delta^H_{1,\alpha}g_\alpha^{-1}\}\Delta^C_{\alpha,\alpha^{-1}}(c)\\
&= (\mu_{C_1} \otimes \mu_{H_\alpha})(I \otimes \tau \otimes I)\{(\sigma_1 \otimes I^H_\alpha)\Delta^H_{1,\alpha}g_\alpha \otimes (\sigma_1 \otimes I^H_\alpha)\Delta^H_{1,\alpha}g_\alpha^{-1}\}\Delta^C_{\alpha,\alpha^{-1}}(c)\\
&= (\mu_{C_1} \otimes \mu_{H_\alpha})(\sigma_1 \otimes \sigma_1 \otimes I^H_\alpha \otimes I^H_\alpha)\\
&\quad (I \otimes \tau \otimes I)(\Delta^H_{1,\alpha} \otimes \Delta^H_{1,\alpha})(g_\alpha \otimes g_\alpha^{-1})\Delta^C_{\alpha,\alpha^{-1}}(c)\\
&= (\mu_{C_1}(\sigma_1 \otimes \sigma_1) \otimes \mu_{H_\alpha})(I \otimes \tau \otimes I)(\Delta^H_{1,\alpha} \otimes \Delta^H_{1,\alpha})(g_\alpha \otimes g_\alpha^{-1})\Delta^C_{\alpha,\alpha^{-1}}(c)\\
&= (\sigma_1\mu_{H_1} \otimes \mu_{H_\alpha})(I \otimes \tau \otimes I)(\Delta^H_{1,\alpha} \otimes \Delta^H_{1,\alpha})(g_\alpha \otimes g_\alpha^{-1})\Delta^C_{\alpha,\alpha^{-1}}(c)\\
&= (\sigma_1 \otimes I^H_\alpha)(\mu_{H_1} \otimes \mu_{H_\alpha})(I \otimes \tau \otimes I)(\Delta^H_{1,\alpha} \otimes \Delta^H_{1,\alpha})(g_\alpha \otimes g_\alpha^{-1})\Delta^C_{\alpha,\alpha^{-1}}(c)\\
&= (\sigma_1 \otimes I^H_\alpha)\Delta^H_{1,\alpha}\mu_{H_\alpha}(g_\alpha \otimes g_\alpha^{-1})\Delta^C_{\alpha,\alpha^{-1}}(c)\\
&= (\sigma_1 \otimes I^H_\alpha)\Delta^H_{1,\alpha}(\epsilon^C(c)1_{H_\alpha})\\
&= \epsilon^C(c)(\sigma_1 \otimes I^H_\alpha)\Delta^H_{1,\alpha}(1_{H_\alpha})\\
&= \epsilon^C(c)(1_{C_1} \otimes 1_{H_\alpha}).
\end{aligned}$$

and

$$\begin{aligned}
\{L_{1,\alpha}g_\alpha * (S^C_1 \otimes g_\alpha^{-1})\tau\Delta^C_{\alpha^{-1},1}\}(c) &= \{(\sigma_1 \otimes I^H_\alpha)\Delta^H_{1,\alpha}g_\alpha * (S^C_1 \otimes g_\alpha^{-1})\tau\Delta^C_{\alpha^{-1},1}\}(c)\\
&= \{(I_1 \otimes g_\alpha)\Delta^C_{1,\alpha} * (S^C_1 \otimes g_\alpha^{-1})\tau\Delta^C_{\alpha^{-1},1}\}(c)\\
&= \mu_{C_1\otimes H_\alpha}\{(I_1 \otimes g_\alpha)\Delta^C_{1,\alpha} \otimes (S^C_1 \otimes g_\alpha^{-1})\tau\Delta^C_{\alpha^{-1},1}\}\Delta^C_{\alpha,\alpha^{-1}}(c)\\
&= (\mu_{C_1} \otimes \mu_{H_\alpha})(I \otimes \tau \otimes I)\{(I_1 \otimes g_\alpha)\Delta^C_{1,\alpha} \otimes\\
&\quad (S^C_1 \otimes g_\alpha^{-1})\tau\Delta^C_{\alpha^{-1},1}\}\Delta^C_{\alpha,\alpha^{-1}}(c)\\
&= (\mu_{C_1} \otimes \mu_{H_\alpha})(I \otimes \tau \otimes I)(I_1 \otimes g_\alpha \otimes S^C_1 \otimes g_\alpha^{-1})\\
&\quad (I_1 \otimes I_\alpha \otimes \tau)(\Delta^C_{1,\alpha} \otimes \Delta^C_{\alpha^{-1},1})\Delta^C_{\alpha,\alpha^{-1}}(c)\\
&= (\mu_{C_1} \otimes \mu_{H_\alpha})(I \otimes \tau \otimes I)(I_1 \otimes g_\alpha \otimes S^C_1 \otimes g_\alpha^{-1})(I_1 \otimes I_\alpha \otimes \tau)\\
&\quad (c^1_{11} \otimes c^\alpha_{121} \otimes c^{\alpha^{-1}}_{122} \otimes c^1_2) \qquad \text{by Lemma 3.4 (1)}\\
&= c^1_{11}S^C_1(c^1_2) \otimes g_\alpha(c^\alpha_{121})g_\alpha^{-1}(c^{\alpha^{-1}}_{122})\\
&= c^1_{11}S^C_1(c^1_2) \otimes \epsilon^C(c^\alpha_{12})1_{H_\alpha}\\
&= \epsilon^C(c^\alpha_{12})c^1_{11}S^C_1(c^1_2) \otimes 1_{H_\alpha}\\
&= c^1_1S^C_1(c^1_2) \otimes 1_{H_\alpha}\\
&= \epsilon^C(c)(1_{C_1} \otimes 1_{H_\alpha}).
\end{aligned}$$

$\square$

Lemma 3.6. *For $h \in H_\alpha$, we have*

$$h = g_\alpha \sigma_\alpha(h_1^\alpha) g_\alpha^{-1} \sigma_{\alpha^{-1}}(h_{21}^{\alpha^{-1}}) h_{22}^\alpha.$$

Proof.

$$\begin{aligned}
h &= \epsilon^H(h_1^1) h_2^\alpha \\
&= \varepsilon^C \sigma_1(h_1^1) h_2^\alpha \\
&= (\mu_\alpha(g_\alpha \otimes g_\alpha^{-1}) \Delta^C_{\alpha,\alpha^{-1}}(\sigma_1(h_1^1))) h_2^\alpha \\
&= (\mu_\alpha(g_\alpha \otimes g_\alpha^{-1})(\sigma_\alpha \otimes \sigma_{\alpha^{-1}}) \Delta^H_{\alpha,\alpha^{-1}}(h_1^1)) h_2^\alpha \\
&= \mu_\alpha(\mu_\alpha(g_\alpha \otimes g_\alpha^{-1})(\sigma_\alpha \otimes \sigma_{\alpha^{-1}}) \otimes I_\alpha)(\Delta^H_{\alpha,\alpha^{-1}} \otimes I_\alpha)\Delta^H_{1,\alpha}(h) \\
&= \mu_\alpha(\mu_\alpha(g_\alpha \otimes g_\alpha^{-1})(\sigma_\alpha \otimes \sigma_{\alpha^{-1}}) \otimes I_\alpha)(I_\alpha \otimes \Delta^H_{\alpha^{-1},\alpha})\Delta^H_{\alpha,1}(h) \\
&= g_\alpha \sigma_\alpha(h_1^\alpha) g_\alpha^{-1} \sigma_{\alpha^{-1}}(h_{21}^{\alpha^{-1}}) h_{22}^\alpha.
\end{aligned}$$

$\square$

Lemma 3.7. *For $h \in H_1, \alpha \in \pi$, we have*

$$\mu_\alpha(g_\alpha^{-1}\sigma_{\alpha^{-1}} \otimes I_\alpha)\Delta^H_{\alpha,\alpha^{-1}}(h) \in B_\alpha.$$

Proof.

$$\begin{aligned}
L_{1,\alpha}\mu_\alpha(g_\alpha^{-1}\sigma_{\alpha^{-1}} \otimes I_\alpha)\Delta^H_{\alpha^{-1},\alpha}(h) &= (\sigma_1 \otimes I^H_\alpha)\Delta^H_{1,\alpha}\mu_\alpha(g_\alpha^{-1}\sigma_{\alpha^{-1}} \otimes I_\alpha)\Delta^H_{\alpha^{-1},\alpha}(h) \\
&= \mu_{C_1\otimes H_\alpha}((\sigma_1 \otimes I^H_\alpha)\Delta^H_{1,\alpha} g_\alpha^{-1}\sigma_{\alpha^{-1}} \otimes (\sigma_1 \otimes I^H_\alpha)\Delta^H_{1,\alpha})\Delta^H_{\alpha^{-1},\alpha}(h) \\
&= \mu_{C_1\otimes H_\alpha}((S_1^C \otimes g_\alpha^{-1})\tau\Delta^C_{\alpha^{-1},1}\sigma_{\alpha^{-1}} \otimes (\sigma_1 \otimes I^H_\alpha)\Delta^H_{1,\alpha})\Delta^H_{\alpha^{-1},\alpha}(h) \\
&= \mu_{C_1\otimes H_\alpha}((S_1^C \otimes g_\alpha^{-1})\tau(\sigma_{\alpha^{-1}} \otimes \sigma_1)\Delta^H_{\alpha^{-1},1} \otimes \\
&\quad (\sigma_1 \otimes I^H_\alpha)\Delta^H_{1,\alpha})\Delta^H_{\alpha^{-1},\alpha}(h) \\
&= \mu_{C_1\otimes H_\alpha}((S_1^C \otimes g_\alpha^{-1})\tau \otimes I_1 \otimes I^H_\alpha) \\
&\quad (\sigma_{\alpha^{-1}} \otimes \sigma_1 \otimes \sigma_1 \otimes I^H_\alpha)(\Delta^H_{\alpha^{-1},1} \otimes \Delta^H_{1,\alpha})\Delta^H_{\alpha^{-1},\alpha}(h) \\
&= \mu_{C_1\otimes H_\alpha}((S_1^C \otimes g_\alpha^{-1})\tau \otimes I_1 \otimes I^H_\alpha) \\
&\quad (\sigma_{\alpha^{-1}}(h_{11}^{\alpha^{-1}}) \otimes \sigma_1(h_{121}^1) \otimes \sigma_1(h_{122}^1) \otimes h_2^\alpha) \\
&= S_1^C(\sigma_1(h_{121}^1))\sigma_1(h_{122}^1) \otimes g_\alpha^{-1}\sigma_{\alpha^{-1}}(h_{11}^{\alpha^{-1}})h_2^\alpha \\
&= \epsilon^C(\sigma_1(h_{12}^1)1 \otimes g_\alpha^{-1}\sigma_{\alpha^{-1}}(h_{11}^{\alpha^{-1}})h_2^\alpha \\
&= \epsilon^H(h_{12}^1)1 \otimes g_\alpha^{-1}\sigma_{\alpha^{-1}}(h_{11}^{\alpha^{-1}})h_2^\alpha \\
&= 1 \otimes g_\alpha^{-1}\sigma_{\alpha^{-1}}(\epsilon^H(h_{12}^1)h_{11}^{\alpha^{-1}})h_2^\alpha \\
&= 1 \otimes g_\alpha^{-1}\sigma_{\alpha^{-1}}(h_1^{\alpha^{-1}})h_2^\alpha \\
&= 1 \otimes \mu_\alpha(g_\alpha^{-1}\sigma_{\alpha^{-1}} \otimes I_\alpha)\Delta^H_{\alpha^{-1},\alpha}(h).
\end{aligned}$$

$\square$

Lemma 3.8. *For $\alpha \in \pi$,*

$$(\sigma_\alpha \otimes \sigma_{\alpha^{-1}} \otimes I^H_\alpha)(\Delta^H_{\alpha,\alpha^{-1}} \otimes I^H_\alpha)\Delta^H_{1,\alpha} g_\alpha = (I_\alpha \otimes I_{\alpha^{-1}} \otimes g_\alpha)(I_\alpha \otimes \Delta^C_{\alpha^{-1},\alpha})\Delta^C_{\alpha,1}$$

Proof.

$$\begin{aligned}(\sigma_\alpha \otimes \sigma_{\alpha^{-1}} \otimes I^H_\alpha)(\Delta^H_{\alpha,\alpha^{-1}} \otimes I^H_\alpha)\Delta^H_{1,\alpha} g_\alpha &= ((\sigma_\alpha \otimes \sigma_{\alpha^{-1}})\Delta^H_{\alpha,\alpha^{-1}} \otimes I^H_\alpha)\Delta^H_{1,\alpha} g_\alpha \\ &= (\Delta^C_{\alpha,\alpha^{-1}}\sigma_1 \otimes I^H_\alpha)\Delta^H_{1,\alpha} g_\alpha \\ &= (\Delta^C_{\alpha,\alpha^{-1}} \otimes I^H_\alpha)(\sigma_1 \otimes I^H_\alpha)\Delta^H_{1,\alpha} g_\alpha \\ &= (\Delta^C_{\alpha,\alpha^{-1}} \otimes I^H_\alpha)(I_1 \otimes g_\alpha)\Delta^C_{1,\alpha} \\ &= (I_\alpha \otimes I_{\alpha^{-1}} \otimes g_\alpha)(\Delta^C_{\alpha,\alpha^{-1}} \otimes I_\alpha)\Delta^C_{1,\alpha} \\ &= (I_\alpha \otimes I_{\alpha^{-1}} \otimes g_\alpha)(I_\alpha \otimes \Delta^C_{\alpha^{-1},\alpha})\Delta^C_{\alpha,1}\end{aligned}$$

□

Lemma 3.9. *For $\alpha \in \pi, b \in B_\alpha$*

$$(\sigma_\alpha \otimes \sigma_{\alpha^{-1}} \otimes I^H_\alpha)(\Delta^H_{\alpha,\alpha^{-1}} \otimes I^H_\alpha)\Delta^H_{1,\alpha}(b) = 1 \otimes 1 \otimes b$$

Proof.

$$\begin{aligned}(\sigma_\alpha \otimes \sigma_{\alpha^{-1}} \otimes I^H_\alpha)(\Delta^H_{\alpha,\alpha^{-1}} \otimes I^H_\alpha)\Delta^H_{1,\alpha}(b) &= ((\sigma_\alpha \otimes \sigma_{\alpha^{-1}})\Delta^H_{\alpha,\alpha^{-1}} \otimes I^H_\alpha)\Delta^H_{1,\alpha}(b) \\ &= (\Delta^C_{\alpha,\alpha^{-1}}\sigma_1 \otimes I^H_\alpha)\Delta^H_{1,\alpha}(b) \\ &= (\Delta^C_{\alpha,\alpha^{-1}} \otimes I^H_\alpha)(\sigma_1 \otimes I^H_\alpha)\Delta^H_{1,\alpha}(b) \\ &= (\Delta^C_{\alpha,\alpha^{-1}} \otimes I^H_\alpha)(1 \otimes b) = 1 \otimes 1 \otimes b\end{aligned}$$

□

Now, we can prove the main theorems in this section.

Theorem 3.10. *H is isomorphic to $C \otimes B$ as vector space.*

Proof. We define $A = \{A_\alpha : C_\alpha \otimes B_\alpha \to H_\alpha\}_{\alpha\in\pi}$ as follow

$$A_\alpha(c \otimes b) = \mu_\alpha(g_\alpha \otimes I_\alpha)(c \otimes b) = g_\alpha(c)b.$$

Clear A_α is linear and by Lemma 3.6 and Lemma 3.7 that A_α is surjective for all $\alpha \in \pi$. We define $A^{-1}_\alpha = (\sigma_\alpha \otimes \mu_\alpha(g^{-1}_\alpha\sigma_{\alpha^{-1}} \otimes I_\alpha))(\Delta^H_{\alpha,\alpha^{-1}} \otimes I_\alpha)\Delta^H_{1,\alpha}$. We'll prove that for $\alpha \in \pi, A_\alpha A^{-1}_\alpha = I_{H_\alpha}$ and $A^{-1}_\alpha A_\alpha = I_{C_\alpha\otimes B_\alpha}$. Firstly, let $h \in H_\alpha$ and $c \otimes b \in C_\alpha \otimes B_\alpha$

$$\begin{aligned}A_\alpha A^{-1}_\alpha(h) &= A_\alpha((\sigma_\alpha \otimes \mu_\alpha(g^{-1}_\alpha\sigma_{\alpha^{-1}} \otimes I_\alpha))(\Delta^H_{\alpha,\alpha^{-1}} \otimes I_\alpha)\Delta^H_{1,\alpha}(h)) \\ &= A_\alpha((\sigma_\alpha \otimes \mu_\alpha(g^{-1}_\alpha\sigma_{\alpha^{-1}} \otimes I_\alpha))(I_\alpha \otimes \Delta^H_{\alpha^{-1},\alpha})\Delta^H_{\alpha,1}(h)) \\ &= A_\alpha(\sigma_\alpha(h^\alpha_1) \otimes g^{-1}_\alpha\sigma_{\alpha^{-1}}(h^{\alpha^{-1}}_{21})h^\alpha_{22}) \\ &= g_\alpha\sigma_\alpha(h^\alpha_1)g^{-1}_\alpha\sigma_{\alpha^{-1}}(h^{\alpha^{-1}}_{21})h^\alpha_{22} \\ &= h \qquad\qquad \text{by Lemma 3.6}\end{aligned}$$

Secondly, since, for $\alpha \in \pi$ we have $((\sigma_\alpha \otimes \sigma_{\alpha^{-1}})\Delta^H_{\alpha,\alpha^{-1}} \otimes I^H_\alpha)\Delta^H_{1,\alpha}$ is an algebra map, then

$$\begin{aligned}
A^{-1}_\alpha A_\alpha(c \otimes b) &= (\sigma_\alpha \otimes \mu_\alpha(g^{-1}_\alpha \sigma_{\alpha^{-1}} \otimes I_\alpha))(\Delta^H_{\alpha,\alpha^{-1}} \otimes I_\alpha)\Delta^H_{1,\alpha}\mu_\alpha(g_\alpha \otimes I_\alpha)(c \otimes b) \\
&= (I_\alpha \otimes \mu_\alpha)(I_\alpha \otimes g^{-1}_\alpha \otimes I_\alpha)(\sigma_\alpha \otimes \sigma_{\alpha^{-1}} \otimes I^H_\alpha)(\Delta^H_{\alpha,\alpha^{-1}} \otimes I^H_\alpha)\Delta^H_{1,\alpha}\mu_\alpha(g_\alpha \otimes I_\alpha)(c \otimes b) \\
&= (I_\alpha \otimes \mu_\alpha)(I_\alpha \otimes g^{-1}_\alpha \otimes I_\alpha)((\sigma_\alpha \otimes \sigma_{\alpha^{-1}})\Delta^H_{\alpha,\alpha^{-1}} \otimes I^H_\alpha) \\
&\quad \Delta^H_{1,\alpha}\mu_\alpha(g_\alpha \otimes I_\alpha)(c \otimes b) \\
&= (I_\alpha \otimes \mu_\alpha)(I_\alpha \otimes g^{-1}_\alpha \otimes I_\alpha)\mu_{H_\alpha \otimes H_{\alpha^{-1}} \otimes H_\alpha} \\
&\quad (((\sigma_\alpha \otimes \sigma_{\alpha^{-1}})\Delta^H_{\alpha,\alpha^{-1}} \otimes I^H_\alpha)\Delta^H_{1,\alpha} g_\alpha \otimes ((\sigma_\alpha \otimes \sigma_{\alpha^{-1}})\Delta^H_{\alpha,\alpha^{-1}} \otimes I^H_\alpha)\Delta^H_{1,\alpha})(c \otimes b) \\
&= (I_\alpha \otimes \mu_\alpha)(I_\alpha \otimes g^{-1}_\alpha \otimes I_\alpha)\mu_{H_\alpha \otimes H_{\alpha^{-1}}} \\
&\quad ((I_\alpha \otimes I_{\alpha^{-1}} \otimes g_\alpha)(I_\alpha \otimes \Delta^C_{\alpha^{-1},\alpha})\Delta^C_{\alpha,1}(c) \otimes 1 \otimes 1 \otimes b) \quad \text{by Lemmas 3.8, 3.9} \\
&= c^\alpha_1 \otimes g^{-1}_\alpha(c^{\alpha^{-1}}_{21})g_\alpha(c^\alpha_{22})b = c^\alpha_1 \otimes \epsilon(c^1_2)b = \epsilon(c^1_2)c^\alpha_1 \otimes b = c \otimes b.
\end{aligned}$$

□

Remark 3.11. In any Hopf π -coalgebra H, every H_α is left H_1 -comodule by $\Delta_{1,\alpha}$. i.e., the following digrams are commute

$$\begin{tikzcd}
H_\alpha \arrow[r, "\Delta_{1,\alpha}"] \arrow[d, "\Delta_{1,\alpha}"'] & H_1 \otimes H_\alpha \arrow[d, "\Delta_{1,1} \otimes I_\alpha"] \\
H_1 \otimes H_\alpha \arrow[r, "I_1 \otimes \Delta_{1,\alpha}"'] & H_1 \otimes H_1 \otimes H_\alpha
\end{tikzcd}
\qquad
\begin{tikzcd}
H_\alpha \arrow[r, "\Delta_{1,\alpha}"] \arrow[dr, "\sim"'] & H_1 \otimes H_\alpha \arrow[d, "\epsilon^H \otimes I_\alpha"] \\
 & K \otimes H_\alpha
\end{tikzcd}$$

Theorem 3.12. *If H have a left cosection, then $Ind(\rho)$ is isomorphic to $V \otimes B = \{(V \otimes B)_\alpha = V_1 \otimes B_\alpha\}_{\alpha \in \pi}$ as right B-module.*

Proof. Firstly, we'll prove that $L_{1,\alpha}\eta_\alpha g_1 = (I_1 \otimes \eta_\alpha g_1)\Delta^C_{1,1}$.

$$\begin{aligned}
L_{1,\alpha}\eta_\alpha g_1 &= (\sigma_1 \otimes I^H_\alpha)\Delta^H_{1,\alpha}\eta_\alpha g_1 \\
&= (\sigma_1 \otimes I^H_\alpha)(I_1 \otimes \eta_\alpha)\Delta^H_{1,1} g_1 \\
&= (I_1 \otimes \eta_\alpha)(\sigma_1 \otimes I^H_1)\Delta^H_{1,1} g_1 \\
&= (I_1 \otimes \eta_\alpha)L_{1,1} g_1 \\
&= (I_1 \otimes \eta_\alpha)(I_1 \otimes g_1)\Delta^C_{1,1} \\
&= (I_1 \otimes \eta_\alpha g_1)\Delta^C_{1,1}.
\end{aligned}$$

Now, we define $T_\alpha = (I_1 \otimes \eta_\alpha g_1)\rho_{1,1} : V_1 \to Ind(\rho)_\alpha$. We'll prove that $T_\alpha(v_1) \in Ind(\rho)_\alpha$.

$$\begin{aligned}
(I_1 \otimes L_{1,\alpha})T_\alpha(v_1) &= (I_1 \otimes L_{1,\alpha})(I_1 \otimes \eta_\alpha g_1)\rho_{1,1}(v_1) \\
&= (I_1 \otimes L_{1,\alpha}\eta_\alpha g_1)\rho_{1,1}(v_1) \\
&= (I_1 \otimes (I_1 \otimes \eta_\alpha g_1)\Delta^C_{1,1})\rho_{1,1}(v_1) \\
&= (I_1 \otimes I_1 \otimes \eta_\alpha g_1)(I_1 \otimes \Delta^C_{1,1})\rho_{1,1}(v_1) \\
&= (I_1 \otimes I_1 \otimes \eta_\alpha g_1)(\rho_{1,1} \otimes I_1)\rho_{1,1}(v_1) \\
&= (\rho_{1,1} \otimes I_\alpha)(I_1 \otimes \eta_\alpha g_1)\rho_{1,1}(v_1) \\
&= (\rho_{1,1} \otimes I_\alpha)T_\alpha(v_1).
\end{aligned}$$

For $\alpha \in \pi$ we define $q_\alpha : V_1 \otimes B_\alpha \to Ind(\rho)_\alpha$ where

$$q_\alpha(v \otimes b) = \lambda_\alpha(T_\alpha(v) \otimes b) = v_1 \otimes \eta_\alpha g_1(v_2)b.$$

Clear $q_\alpha(v \otimes b) \in Ind(\rho)_\alpha$ and q_α is linear. We define

$$q_\alpha^{-1} : Ind(\rho)_\alpha \to V_1 \otimes B_\alpha$$

where

$$\begin{aligned}
q_\alpha^{-1} &= (I_1 \otimes \mu_\alpha(\eta_\alpha g_1^{-1} \otimes I_\alpha))(\rho_{1,1} \otimes I_\alpha) && (3.3) \\
&= (I_1 \otimes \mu_\alpha(\eta_\alpha g_1^{-1} \otimes I_\alpha))(I_1 \otimes L_{1,\alpha}) && (3.4)
\end{aligned}$$

$$i.e., q_\alpha^{-1}(v \otimes h) = v_1 \otimes \eta_\alpha g_1^{-1}(v_2)h = v \otimes \eta_\alpha g_1^{-1}\sigma_1(h_1^1)h_2^\alpha.$$

We'll prove that $q_\alpha^{-1}(v \otimes h) \in V_1 \otimes B_\alpha$.

$$\begin{aligned}
(I_1 \otimes L_{1,\alpha})q_\alpha^{-1}(v \otimes h) &= (I_1 \otimes L_{1,\alpha})(I_1 \otimes \mu_\alpha(\eta_\alpha g_1^{-1} \otimes I_\alpha))(I_1 \otimes L_{1,\alpha})(v \otimes h) \quad \text{by Equation 3.4} \\
&= (I_1 \otimes \mu_{C_1 \otimes H_\alpha}(L_{1,\alpha}\eta_\alpha g_1^{-1} \otimes L_{1,\alpha}))(I_1 \otimes L_{1,\alpha})(v \otimes h) \\
&= v \otimes \mu_{C_1 \otimes H_\alpha}((I_1 \otimes \eta_\alpha)L_{1,1}g_1^{-1} \otimes L_{1,\alpha})L_{1,\alpha}(h) \\
&= v \otimes \mu_{C_1 \otimes H_\alpha}((I_1 \otimes \eta_\alpha)(S_1^C \otimes g_1^{-1})\tau\Delta^H_{1,1} \otimes L_{1,\alpha})L_{1,\alpha}(h) \quad \text{by Lemma 3.4} \\
&= v \otimes \mu_{C_1 \otimes H_\alpha}((S_1^C \otimes \eta_\alpha g_1^{-1})\tau\Delta^H_{1,1} \otimes (\sigma_1 \otimes I^H_\alpha)\Delta^H_{1,\alpha})(\sigma_1 \otimes I^H_\alpha)\Delta^H_{1,\alpha}(h) \\
&= v \otimes \mu_{C_1 \otimes H_\alpha}((S_1^C \otimes \eta_\alpha g_1^{-1})\tau \otimes \sigma_1 \otimes I^H_\alpha)(\Delta^H_{1,1} \otimes \Delta^H_{1,\alpha})(\sigma_1 \otimes I^H_\alpha)\Delta^H_{1,\alpha}(h) \\
&= v \otimes \mu_{C_1 \otimes H_\alpha}((S_1^C \otimes \eta_\alpha g_1^{-1})\tau \otimes \sigma_1 \otimes I^H_\alpha) \\
&\quad\ (\sigma_1 \otimes \sigma_1 \otimes I^H_1 \otimes I^H_\alpha)(\Delta^H_{1,1} \otimes \Delta^H_{1,\alpha})\Delta^H_{1,\alpha}(h) \\
&= v \otimes S_1^C\sigma_1(h^1_{121})\sigma_1(h^1_{122}) \otimes \eta_\alpha g_1^{-1}\sigma_1(h^1_{11}))h_2^\alpha \\
&= v \otimes \epsilon^C\sigma_1(h^1_{12})1 \otimes \eta_\alpha g_\alpha^{-1}\sigma_1(h^1_{11}))h_2^\alpha \\
&= v \otimes \epsilon^H(h^1_{12})1 \otimes \eta_\alpha g_1^{-1}\sigma_1(h^1_{11}))h_2^\alpha \\
&= v \otimes 1 \otimes \eta_\alpha g_1^{-1}\sigma_1(\epsilon^H(h^1_{12})h^1_{11}))h_2^\alpha \\
&= v \otimes 1 \otimes \eta_\alpha g_1^{-1}\sigma_1(h_1^1)h_2^\alpha \\
&= v \otimes 1 \otimes \mu_\alpha(\eta_\alpha g_1^{-1} \otimes I_\alpha))L_{1,\alpha}(h)
\end{aligned}$$

Now, we will prove $q_\alpha q_\alpha^{-1} = I$ and $q_\alpha^{-1} q_\alpha = I$.

$$\begin{aligned}
q_\alpha q_\alpha^{-1}(v \otimes h) &= (I_1 \otimes \mu_\alpha)(I_1 \otimes \eta_\alpha g_1 \otimes I_\alpha)(\rho_{1,1} \otimes I_\alpha) \\
&\quad (I_1 \otimes \mu_\alpha(\eta_\alpha g_1^{-1} \otimes I_\alpha))(\rho_{1,1} \otimes I_\alpha)(v \otimes h) \quad \text{by Equation 3.4} \\
&= (I_1 \otimes \mu_\alpha)(I_1 \otimes \eta_\alpha g_1 \otimes I_\alpha)(I_1 \otimes I_1 \otimes \mu_\alpha(\eta_\alpha g_1^{-1} \otimes I_\alpha)) \\
&\quad (\rho_{1,1} \otimes I_1 \otimes I_\alpha)(\rho_{1,1} \otimes I_\alpha)(v \otimes h) \\
&= (I_1 \otimes \mu_\alpha)(I_1 \otimes \eta_\alpha g_1 \otimes I_\alpha)(I_1 \otimes I_1 \otimes \mu_\alpha(\eta_\alpha g_1^{-1} \otimes I_\alpha)) \\
&\quad ((\rho_{1,1} \otimes I_1)\rho_{1,1} \otimes I_\alpha)(v \otimes h) \\
&= (I_1 \otimes \mu_\alpha)(I_1 \otimes \eta_\alpha g_1 \otimes I_\alpha)(I_1 \otimes I_1 \otimes \mu_\alpha(\eta_\alpha g_1^{-1} \otimes I_\alpha)) \\
&\quad ((I_1 \otimes \Delta^C_{1,1} \otimes I_\alpha)(\rho_{1,1} \otimes I_\alpha)(v \otimes h) \\
&= v_1^1 \otimes \eta_\alpha g_1(v_{21}^1)\eta_\alpha g_1^{-1}(v_{22}^1)h \\
&= v_1^1 \otimes \eta_\alpha(g_1(v_{21}^1)g_1^{-1}(v_{22}^1))h \\
&= v_1^1 \otimes \eta_\alpha(\epsilon(v_2^1)1)h \\
&= \epsilon(v_2^1)v_1^1 \otimes \eta_\alpha(1)h \\
&= v \otimes h
\end{aligned}$$

Also,

$$\begin{aligned}
q_\alpha^{-1} q_\alpha(v \otimes b) &= (I_1 \otimes \mu_\alpha(\eta_\alpha g_1^{-1} \otimes I_\alpha))(I_1 \otimes L_{1,\alpha})(I_1 \otimes \mu_\alpha) \\
&\quad (I_1 \otimes \eta_\alpha g_1 \otimes I_\alpha)(\rho_{1,1} \otimes I_\alpha)(v \otimes b) \quad \text{by Equation 3.4} \\
&= (I_1 \otimes \mu_\alpha(\eta_\alpha g_1^{-1} \otimes I_\alpha))(I_1 \otimes L_{1,\alpha})(v_1 \otimes \eta_\alpha g_1(v_2^1)b) \\
&= (I_1 \otimes \mu_\alpha(\eta_\alpha g_1^{-1} \otimes I_\alpha))(v_1 \otimes L_{1,\alpha}(\eta_\alpha g_1(v_2^1))L_{1,\alpha}(b)) \\
&= (I_1 \otimes \mu_\alpha(\eta_\alpha g_1^{-1} \otimes I_\alpha))(v_1 \otimes (I_1 \otimes \eta_\alpha)L_{1,1}g_1(v_2^1) \cdot (1 \otimes b)) \\
&= (I_1 \otimes \mu_\alpha(\eta_\alpha g_1^{-1} \otimes I_\alpha))(v_1 \otimes (I_1 \otimes \eta_\alpha)(I_1 \otimes g_1)\Delta^C_{1,1}(v_2^1) \cdot (1 \otimes b)) \\
&= (I_1 \otimes \mu_\alpha(\eta_\alpha g_1^{-1} \otimes I_\alpha))(v_1 \otimes v_{21}^1 \otimes \eta_\alpha g_1(v_{22}^1)b) \\
&= v_1 \otimes \eta_\alpha g_1^{-1}(v_{21}^1)\eta_\alpha g_1(v_{22}^1)b \\
&= v_1 \otimes \eta_\alpha(g_1^{-1}(v_{21}^1)g_1(v_{22}^1))b \\
&= v_1 \otimes \eta_\alpha(\epsilon^C(v_2^1)1)b \\
&= \epsilon^C(v_2^1)v_1 \otimes \eta_\alpha(1)b = v \otimes b.
\end{aligned}$$

Now we'll prove that q_α is module map for all $\alpha \in \pi$,

$$\begin{aligned}
q_\alpha(I_1 \otimes \mu_\alpha)(v \otimes b \otimes k) &= q_\alpha(v \otimes bk) \\
&= \lambda_\alpha(T_\alpha(v) \otimes bk) \\
&= \lambda_\alpha((I_1 \otimes \eta_\alpha g_1)\rho_{1,1}(v) \otimes bk)
\end{aligned}$$

$$
\begin{aligned}
&= \lambda_\alpha(v_1 \otimes \eta_\alpha g_1(v_2) \otimes bk) \\
&= v_1 \otimes \eta_\alpha g_1(v_2) bk \\
&= v_1 \otimes (\eta_\alpha g_1(v_2) b) k \\
&= \lambda_\alpha(v_1 \otimes (\eta_\alpha g_1(v_2) b) \otimes k) \\
&= \lambda_\alpha(q_\alpha \otimes I_\alpha)(v \otimes b \otimes k).
\end{aligned}
$$

□

Throughout this last part (C, σ) is left $\pi-$ coisotropic quantum subgroup of H and $V = \{V_\alpha\}_{\alpha \in \pi}$ be a right $\pi-$comodule over C and $G = \{G_\alpha\}_{\alpha \in \pi}$,where

$$G_\alpha = \{h \in H_\alpha : L_{1,\alpha}(h) = (\sigma_1 \otimes I_\alpha^H)\Delta_{1,\alpha}^H(h) = \sigma_1(1) \otimes h\}.$$

Lemma 3.13. *$Ind(\rho)$ is right G-module*

Proof. Similar to Lemma 3.1. □

Lemma 3.14. *For $\alpha \in \pi, u \in H_1, c \in C_{\alpha^{-1}}$ and $v \in \sigma_{\alpha^{-1}}^{-1}(c)$ we have*

$$
\begin{aligned}
&(\sigma_1 \otimes I_\alpha^H)(\mu_1 \otimes I_\alpha)(I_1 \otimes \tau)(\Delta_{1,\alpha}^H g_\alpha^{-1} \otimes I_1)(c \otimes u) \\
&= (I_1 \otimes g_\alpha^{-1})(\sigma_1 \otimes \sigma_{\alpha^{-1}})(\mu_1(S_1^H \otimes I_1) \otimes I_{\alpha^{-1}})(I_1 \otimes \tau)(\tau\Delta_{\alpha^{-1},1}^H \otimes I_1)(v \otimes u)
\end{aligned}
$$

Proof. It can be proved with usual Hopf $\pi-$coalgebra techniques. □

Theorem 3.15. *H is isomorphic to $C \otimes G$ as vector space.*

Proof.

1. For $h \in H_\alpha$, as in Lemma 3.6, $h = g_\alpha \sigma_\alpha(h_1^\alpha) g_\alpha^{-1} \sigma_{\alpha^{-1}}(h_{21}^{\alpha^{-1}}) h_{22}^\alpha$.
2. For $h \in H_1, \alpha \in \pi$, we'll prove that

$$\mu_\alpha(g_\alpha^{-1}\sigma_{\alpha^{-1}} \otimes I_\alpha)\Delta_{\alpha^{-1},\alpha}^H(h) \in G_\alpha.$$

as follow

$$
\begin{aligned}
L_{1,\alpha}\mu_\alpha(g_\alpha^{-1}\sigma_{\alpha^{-1}}\otimes I_\alpha)\Delta^H_{\alpha^{-1},\alpha}(h) &= (\sigma_1\otimes I^H_\alpha)\Delta^H_{1,\alpha}\mu_\alpha(g_\alpha^{-1}\sigma_{\alpha^{-1}}\otimes I_\alpha)\Delta^H_{\alpha^{-1},\alpha}(h)\\
&= (\sigma_1\otimes I^H_\alpha)\mu_{H_1\otimes H_\alpha}(\Delta^H_{1,\alpha}\otimes\Delta^H_{1,\alpha})(g_\alpha^{-1}\sigma_{\alpha^{-1}}\otimes I^H_\alpha)\Delta^H_{\alpha^{-1},\alpha}(h)\\
&= (\sigma_1\mu_{H_1}\otimes\mu_{H_\alpha})(I\otimes\tau\otimes I)(\Delta^H_{1,\alpha}\otimes\Delta^H_{1,\alpha})\\
&\quad (g_\alpha^{-1}\sigma_{\alpha^{-1}}\otimes I^H_\alpha)\Delta^H_{\alpha^{-1},\alpha}(h)\\
&= (I_1\otimes\mu_{H_\alpha})((\sigma_1\mu_{H_1}\otimes I_\alpha)(I\otimes\tau)\otimes I^H_\alpha)\\
&\quad (\Delta^H_{1,\alpha}g_\alpha^{-1}\sigma_{\alpha^{-1}}\otimes I_1\otimes I^H_\alpha)(I_{\alpha^{-1}}\otimes\Delta^H_{1,\alpha})\Delta^H_{\alpha^{-1},\alpha}(h)\\
&= (I_1\otimes\mu_{H_\alpha})((\sigma_1\mu_{H_1}\otimes I_\alpha)(I\otimes\tau)(\Delta^H_{1,\alpha}g_\alpha^{-1}\sigma_{\alpha^{-1}}\otimes I_1)\otimes I^H_\alpha)\\
&\quad (I_{\alpha^{-1}}\otimes\Delta^H_{1,\alpha})\Delta^H_{\alpha^{-1},\alpha}(h)\\
&= (I_1\otimes\mu_{H_\alpha})((I_1\otimes g_\alpha^{-1})(\sigma_1\otimes\sigma_{\alpha^{-1}})(\mu_1(S^H_1\otimes I_1)\otimes I_{\alpha^{-1}})\\
&\quad (I_1\otimes\tau)(\tau\Delta^H_{\alpha^{-1},1}\otimes I_1)\otimes I^H_\alpha)(I_{\alpha^{-1}}\otimes\Delta^H_{1,\alpha})\Delta^H_{\alpha^{-1},\alpha}(h)\\
&= (I_1\otimes\mu_{H_\alpha})((I_1\otimes g_\alpha^{-1})(\sigma_1\otimes\sigma_{\alpha^{-1}})(\mu_1(S^H_1\otimes I_1)\otimes I_{\alpha^{-1}})\\
&\quad (I_1\otimes\tau)(\tau\otimes I_1)\otimes I^H_\alpha)(\Delta^H_{\alpha^{-1},1}\otimes\Delta^H_{1,\alpha})\Delta^H_{\alpha^{-1},\alpha}(h)\\
&= (I_1\otimes\mu_{H_\alpha})[(\sigma_1\otimes g_\alpha^{-1}\sigma_{\alpha^{-1}})(\mu_1(S^H_1\otimes I_1)\otimes I_{\alpha^{-1}})\\
&\quad (I_1\otimes\tau)(\tau\otimes I_1)\otimes I^H_\alpha](h_{11}^{\alpha^{-1}}\otimes h^1_{121}\otimes h^1_{122}\otimes h^\alpha_2)\\
&= \sigma_1(S^H_1(h^1_{121})h^1_{122})\otimes g_\alpha^{-1}(\sigma_{\alpha^{-1}}(h_{11}^{\alpha^{-1}}))h^\alpha_2\\
&= \sigma_1(\epsilon^H(h^1_{12})1)\otimes g_\alpha^{-1}(\sigma_{\alpha^{-1}}(h_{11}^{\alpha^{-1}}))h^\alpha_2\\
&= \sigma_1(1)\otimes g_\alpha^{-1}(\sigma_{\alpha^{-1}}(\epsilon^H(h^1_{12})h_{11}^{\alpha^{-1}}))h^\alpha_2\\
&= \sigma_1(1)\otimes g_\alpha^{-1}(\sigma_{\alpha^{-1}}(h_1^{\alpha^{-1}}))h^\alpha_2\\
&= \sigma_1(1)\otimes\mu_\alpha(g_\alpha^{-1}\sigma_{\alpha^{-1}}\otimes I_\alpha)\Delta^H_{\alpha^{-1},\alpha}(h)
\end{aligned}
$$

3. We define $A_\alpha : C_\alpha\otimes G_\alpha\to H_\alpha$ as follow

$$A_\alpha(c\otimes b)=\mu_\alpha(f_\alpha\otimes I_\alpha)(c\otimes b)=f_\alpha(c)b.$$

Clear A_α is linear. We define $A_\alpha^{-1} : H_\alpha\to C_\alpha\otimes K_\alpha$ as

$$A_\alpha^{-1}(h)=(\sigma_\alpha\otimes\mu_\alpha(g_\alpha^{-1}\sigma_{\alpha^{-1}}\otimes I_\alpha))(\Delta^H_{\alpha,\alpha^{-1}}\otimes I_\alpha)\Delta^H_{1,\alpha}(h).$$

We‘ll prove that for $\alpha\in\pi, A_\alpha A_\alpha^{-1}=I_{H_\alpha}$ and $A_\alpha^{-1}A_\alpha=I_{C_\alpha\otimes B_\alpha}$. Let $h\in H_\alpha, c\otimes b\in C_\alpha\otimes G_\alpha$,

$$
\begin{aligned}
A_\alpha A_\alpha^{-1}(h) &= A_\alpha((\sigma_\alpha\otimes\mu_\alpha(g_\alpha^{-1}\sigma_{\alpha^{-1}}\otimes I_\alpha))(\Delta^H_{\alpha,\alpha^{-1}}\otimes I_\alpha)\Delta^H_{1,\alpha}(h))\\
&= A_\alpha((\sigma_\alpha\otimes\mu_\alpha(g_\alpha^{-1}\sigma_{\alpha^{-1}}\otimes I_\alpha))(I_\alpha\otimes\Delta^H_{\alpha^{-1},\alpha})\Delta^H_{\alpha,1}(h))\\
&= A_\alpha(\sigma_\alpha(h^\alpha_1)\otimes g_\alpha^{-1}\sigma_{\alpha^{-1}}(h_{21}^{\alpha^{-1}})h^\alpha_{22})\\
&= g_\alpha\sigma_\alpha(h^\alpha_1)g_\alpha^{-1}\sigma_{\alpha^{-1}}(h_{21}^{\alpha^{-1}})h^\alpha_{22}=h.
\end{aligned}
$$

Also,

$$
\begin{aligned}
A_\alpha^{-1}A_\alpha(c\otimes k) &= (\sigma_\alpha\otimes\mu_\alpha(g_\alpha^{-1}\sigma_{\alpha^{-1}}\otimes I_\alpha))(\Delta^H_{\alpha,\alpha^{-1}}\otimes I_\alpha)\Delta^H_{1,\alpha}\mu_\alpha(g_\alpha\otimes I_\alpha)(c\otimes k)\\
&= (I_\alpha\otimes\mu_\alpha)(I_\alpha\otimes g_\alpha^{-1}\otimes I_\alpha)(\sigma_\alpha\otimes\sigma_{\alpha^{-1}}\otimes I^H_\alpha)(\Delta^H_{\alpha,\alpha^{-1}}\otimes I^H_\alpha)\Delta^H_{1,\alpha}\mu_\alpha(g_\alpha\otimes I_\alpha)(c\otimes k)\\
&= (I_\alpha\otimes\mu_\alpha)(I_\alpha\otimes g_\alpha^{-1}\otimes I_\alpha)((\sigma_\alpha\otimes\sigma_{\alpha^{-1}})\Delta^H_{\alpha,\alpha^{-1}}\otimes I^H_\alpha)\Delta^H_{1,\alpha}\mu_\alpha(g_\alpha\otimes I_\alpha)(c\otimes k)\\
&= (I_\alpha\otimes\mu_\alpha)(I_\alpha\otimes g_\alpha^{-1}\otimes I_\alpha)(\Delta^C_{\alpha,\alpha^{-1}}\sigma_1\otimes I^H_\alpha)\mu_{H_\alpha\otimes H_{\alpha^{-1}}}(\Delta^H_{1,\alpha}g_\alpha\otimes\Delta^H_{1,\alpha})(c\otimes k)\\
&= (I_\alpha\otimes\mu_\alpha)(I_\alpha\otimes g_\alpha^{-1}\otimes I_\alpha)(\Delta^C_{\alpha,\alpha^{-1}}\otimes I^H_\alpha)(\sigma_1\mu_1\otimes\mu_\alpha)\\
&\quad (I\otimes\tau\otimes I)(\Delta^H_{1,\alpha}g_\alpha\otimes\Delta^H_{1,\alpha})(c\otimes k)\\
&= (I_\alpha\otimes\mu_\alpha)(I_\alpha\otimes g_\alpha^{-1}\otimes I_\alpha)(\Delta^C_{\alpha,\alpha^{-1}}\otimes\mu_\alpha)\\
&\quad [(\sigma_1\mu_1\otimes I^H_\alpha)(I\otimes\tau)(\Delta^H_{1,\alpha}g_\alpha(c)\otimes k_1^1\otimes k_2^\alpha)]\\
&= (I_\alpha\otimes\mu_\alpha)(I_\alpha\otimes g_\alpha^{-1}\otimes I_\alpha)(\Delta^C_{\alpha,\alpha^{-1}}\otimes\mu_\alpha)\\
&\quad [(I_1\otimes g_\alpha)(\sigma_1\otimes\sigma_\alpha)(\mu_1\otimes I_\alpha)(I_1\otimes\tau)(\Delta^C_{1,\alpha}\otimes I_1)(v\otimes k_1^1)\otimes k_2^\alpha]\\
&= (I_\alpha\otimes\mu_\alpha)(I_\alpha\otimes g_\alpha^{-1}\otimes I_\alpha)(\Delta^C_{\alpha,\alpha^{-1}}\otimes\mu_\alpha)[\sigma_1(v_1^1k_1^1)\otimes g_\alpha\sigma_\alpha(v_2^\alpha)\otimes k_2^\alpha]\\
&= (I_\alpha\otimes\mu_\alpha)(I_\alpha\otimes g_\alpha^{-1}\otimes I_\alpha)(\Delta^C_{\alpha,\alpha^{-1}}\otimes\mu_\alpha)[\omega_1(v_1^1\otimes\sigma_1(k_1^1))\otimes g_\alpha\sigma_\alpha(v_2^\alpha)\otimes k_2^\alpha]\\
&= (I_\alpha\otimes\mu_\alpha)(I_\alpha\otimes g_\alpha^{-1}\otimes I_\alpha)(\Delta^C_{\alpha,\alpha^{-1}}\otimes\mu_\alpha)[\omega_1(v_1^1\otimes\sigma_1(1))\otimes g_\alpha\sigma_\alpha(v_2^\alpha)\otimes k]\\
&= (I_\alpha\otimes\mu_\alpha)(I_\alpha\otimes g_\alpha^{-1}\otimes I_\alpha)(\Delta^C_{\alpha,\alpha^{-1}}\otimes\mu_\alpha)[\sigma_1(v_1^1)\otimes g_\alpha\sigma_\alpha(v_2^\alpha)\otimes k]\\
&= (I_\alpha\otimes\mu_\alpha)[(I_\alpha\otimes g_\alpha^{-1})\Delta^C_{\alpha,\alpha^{-1}}\sigma_1(v_1^1)\otimes g_\alpha\sigma_\alpha(v_2^\alpha)k]\\
&= (I_\alpha\otimes\mu_\alpha)[(I_\alpha\otimes g_\alpha^{-1})(\sigma_\alpha\otimes\sigma_{\alpha^{-1}})\Delta^H_{\alpha,\alpha^{-1}}(v_1^1)\otimes g_\alpha\sigma_\alpha(v_2^\alpha)k]\\
&= (I_\alpha\otimes\mu_\alpha)[(I_\alpha\otimes g_\alpha^{-1})(\sigma_\alpha(v_1^\alpha)\otimes\sigma_{\alpha^{-1}}(v_2^{\alpha^{-1}})\otimes g_\alpha\sigma_\alpha(v_3^\alpha)k]\\
&= \sigma_\alpha(v_1^\alpha)\otimes g_\alpha^{-1}\sigma_{\alpha^{-1}}(v_2^{\alpha^{-1}})g_\alpha\sigma_\alpha(v_3^\alpha)k\\
&= \sigma_\alpha(v_1^\alpha)\otimes\epsilon^C\sigma_1(v_2^1)k\\
&= \sigma_\alpha(\epsilon^H(v_2^1)v_1^\alpha)\otimes k\\
&= \sigma_\alpha(v)\otimes k\\
&= c\otimes k.
\end{aligned}
$$

□

4 Coinduced representations of Hopf group coalgebra

In this section, we study coinduced representation from left $\pi-$coisotropic quantum subgroup. We restrict our attention to finite dimensional case of Hopf $\pi-$coalgebra, i.e., dim $H_\alpha = n^\alpha \prec \infty$ for all $\alpha\in\pi$. Let us start now from left $\pi-$corepresentation $\rho = \{\rho_{\alpha,\beta}\}_{\alpha,\beta\pi}$ of left $\pi-$coisotropic quantum

subgroup (C,σ) on $V=\{V_\alpha\}_{\alpha\in\pi}$. We define $W=\{W_\alpha\}_{\alpha\in\pi}$, where $W_\alpha=\{F_\alpha\in Hom(V_1,H_\alpha):\ L_{1,\alpha}F_\alpha=(I_{C_1}\otimes F_\alpha)\rho_{1,1}\}$ and $L_{\alpha,\beta}=(\sigma_\alpha\otimes I_\beta)\Delta_{\alpha,\beta}$

Lemma 4.1. *For $\alpha,\beta,\gamma\in\pi$,*

$$(L_{\alpha,\beta}\otimes I_\gamma)\Delta_{\alpha\beta,\gamma}=(I_\alpha\otimes\Delta_{\beta,\gamma})L_{\alpha,\beta\gamma}$$

Proof. Similar to Lemma 2.4 . □

Lemma 4.2. *Suppose that $\{e_i^\alpha\}_{i=1}^{n^\alpha}$ and $\{g_i^\alpha\}_{i=1}^{n^\alpha}$ is its dual basis of H_α and H_α^* for all $\alpha\in\pi$. For $\alpha,\beta,\gamma\in\pi$, fix $F_{\alpha\beta\gamma}\in W_{\alpha\beta\gamma}$, if we define the two maps $\xi_1:V_1\to H_\alpha\otimes H_\beta\otimes H_\gamma$ and $\xi_2:V_1\to H_\alpha\otimes H_\beta\otimes H_\gamma$ such that*

$$\xi_1(v_1)=\sum_{i=1}^{n^{\beta\gamma}}\sim(I_\alpha\otimes g_i^{\beta\gamma})\Delta_{\alpha,\beta\gamma}F_{\alpha\beta\gamma}(v_1)\otimes\Delta_{\beta,\gamma}(e_i^{\beta\gamma})$$

$$\xi_2(v_1)=\sum_{h=1}^{n^\beta}\sum_{l=1}^{n^\gamma}[\sim(\sim\otimes I_K)(I_\alpha\otimes(g_h^\beta\otimes g_l^\gamma)\Delta_{\beta,\gamma}]\Delta_{\alpha,\beta\gamma}F_{\alpha\beta\gamma}(v_1)\otimes e_h^\beta\otimes e_l^\gamma$$

then $\xi_1=\xi_2$.

Proof. We put

$$F_{\alpha\beta\gamma}(v_1)=h_{\alpha\beta\gamma}=\sum_{j=1}^{n^{\alpha\beta\gamma}}\lambda_je_j^{\alpha\beta\gamma}$$

$$\Delta_{\beta,\gamma}(e_i^{\beta\gamma})=\sum_{r=1}^{n^\beta}\sum_{s=1}^{n^\gamma}\eta_{rs}^ie_r^\beta\otimes e_s^\gamma$$

$$\Delta_{\alpha,\beta\gamma}(e_i^{\alpha\beta\gamma})=\sum_{r=1}^{n^\alpha}\sum_{s=1}^{n^{\beta\gamma}}\theta_{rs}^ie_r^\alpha\otimes e_s^{\beta\gamma}$$

then we have

$$\begin{aligned}\Delta_{\alpha,\beta\gamma}F_{\alpha\beta\gamma}(v_1)&=\Delta_{\alpha,\beta\gamma}(h_{\alpha\beta\gamma})\\&=\sum_{j=1}^{n^\gamma}\lambda_j\Delta_{\alpha,\beta\gamma}(e_j^{\alpha\beta\gamma})\\&=\sum_{j=1}^{n^\gamma}\sum_{r=1}^{n^\alpha}\sum_{s=1}^{n^{\beta\gamma}}\lambda_j\theta_{rs}^je_r^\alpha\otimes e_s^{\beta\gamma}\end{aligned}$$

imply that

$$\sum_{i=1}^{n^{\beta\gamma}} \sim (I_\alpha \otimes g_i^{\beta\gamma})\Delta_{\alpha,\beta\gamma}F_{\alpha\beta\gamma}(v_1) \otimes \Delta_{\beta,\gamma}(e_i^{\beta\gamma})$$

$$= \sum_{i=1}^{n^{\beta\gamma}}[\sim (I_\alpha \otimes g_i^{\beta\gamma}) \sum_{j=1}^{n^\gamma}\sum_{r=1}^{n^\alpha}\sum_{s=1}^{n^{\beta\gamma}} \lambda_j \theta_{rs}^j e_r^\alpha \otimes e_s^{\beta\gamma}] \otimes \sum_{h=1}^{n^\beta}\sum_{l=1}^{n^\gamma} \eta_{hl}^i e_h^\beta \otimes e_l^\gamma$$

$$= \sum_{i=1}^{n^{\beta\gamma}}\sum_{j=1}^{n^\gamma}\sum_{r=1}^{n^\alpha}\sum_{s=1}^{n^{\beta\gamma}} \lambda_j \theta_{rs}^j g_i^{\beta\gamma}(e_s^{\beta\gamma}) e_r^\alpha \otimes \sum_{h=1}^{n^\beta}\sum_{l=1}^{n^\gamma} \eta_{hl}^i e_h^\beta \otimes e_l^\gamma$$

$$= \sum_{i=1}^{n^{\beta\gamma}}\sum_{j=1}^{n^\gamma}\sum_{r=1}^{n^\alpha} \lambda_j \theta_{ri}^j e_r^\alpha \otimes \sum_{h=1}^{n^\beta}\sum_{l=1}^{n^\gamma} \eta_{hl}^i e_h^\beta \otimes e_l^\gamma$$

$$= \sum_{i=1}^{n^{\beta\gamma}}\sum_{j=1}^{n^\gamma}\sum_{r=1}^{n^\alpha}\sum_{h=1}^{n^\beta}\sum_{l=1}^{n^\gamma} \eta_{hl}^i \lambda_j \theta_{ri}^j e_r^\alpha \otimes e_h^\beta \otimes e_l^\gamma$$

Also

$$\sum_{h=1}^{n^\beta}\sum_{l=1}^{n^\gamma}[\sim (\sim \otimes I_K)(I_\alpha \otimes (g_h^\beta \otimes g_l^\gamma)\Delta_{\beta,\gamma}]\Delta_{\alpha,\beta\gamma}F_{\alpha\beta\gamma}(v_1) \otimes e_h^\beta \otimes e_l^\gamma$$

$$= \sum_{h=1}^{n^\beta}\sum_{l=1}^{n^\gamma}[\sim (\sim \otimes I_K)(I_\alpha \otimes (g_h^\beta \otimes g_l^\gamma)\Delta_{\beta,\gamma}]$$

$$[\sum_{j=1}^{n^\gamma}\sum_{r=1}^{n^\alpha}\sum_{i=1}^{n^{\beta\gamma}} \lambda_j \theta_{ri}^j e_r^\alpha \otimes e_i^{\beta\gamma}] \otimes e_h^\beta \otimes e_l^\gamma$$

$$= \sum_{h=1}^{n^\beta}\sum_{l=1}^{n^\gamma}\sum_{j=1}^{n^\gamma}\sum_{r=1}^{n^\alpha}\sum_{i=1}^{n^{\beta\gamma}} \lambda_j \theta_{ri}^j \sim (\sim \otimes I_K)[e_r^\alpha \otimes (g_h^\beta \otimes g_l^\gamma)\Delta_{\beta,\gamma}(e_i^{\beta\gamma})] \otimes e_h^\beta \otimes e_l^\gamma$$

$$= \sum_{h=1}^{n^\beta}\sum_{l=1}^{n^\gamma}\sum_{j=1}^{n^\gamma}\sum_{r=1}^{n^\alpha}\sum_{i=1}^{n^{\beta\gamma}}\sum_{q=1}^{n^\beta}\sum_{s=1}^{n^\gamma} \lambda_j \theta_{ri}^j \eta_{qs}^i g_h^\beta(e_q^\beta) g_l^\gamma(e_s^\gamma) e_r^\alpha \otimes e_h^\beta \otimes e_l^\gamma$$

$$= \sum_{h=1}^{n^\beta}\sum_{l=1}^{n^\gamma}\sum_{j=1}^{n^\gamma}\sum_{r=1}^{n^\alpha}\sum_{i=1}^{n^{\beta\gamma}} \eta_{hl}^i \lambda_j \theta_{ri}^j e_r^\alpha \otimes e_h^\beta \otimes e_l^\gamma$$

therefore, we have

$$\begin{aligned}
&\sum_{i=1}^{n^{\beta\gamma}} \sim (I_\alpha \otimes g_i^{\beta\gamma})\Delta_{\alpha,\beta\gamma}F_{\alpha\beta\gamma}(v_1) \otimes \Delta_{\beta,\gamma}(e_i^{\beta\gamma}) \\
&\qquad = \sum_{h=1}^{n^\beta}\sum_{l=1}^{n^\gamma}[\sim (\sim \otimes I_K)(I_\alpha \otimes (g_h^\beta \otimes g_l^\gamma)\Delta_{\beta,\gamma}]\Delta_{\alpha,\beta\gamma}F_{\alpha\beta\gamma}(v_1) \otimes e_h^\beta \otimes e_l^\gamma
\end{aligned}$$

then $\xi_1 = \xi_2$ $\square$

Theorem 4.3. $W = \{W_\alpha\}_{\alpha\in\pi}$ *is right* $\pi-$*comodule over* H *by* $\Omega = \{\Omega_{\alpha,\beta}\}_{\alpha,\beta\in\pi}$ *where*

$$\Omega_{\alpha,\beta} : W_{\alpha\beta} \to W_\alpha \otimes H_\beta \;\; \textit{as} \;\; \Omega_{\alpha,\beta}(F_{\alpha\beta}) = \sim (I_\alpha \otimes g^\beta)\Delta_{\alpha,\beta}F_{\alpha\beta} \otimes e^\beta.$$

Proof. Firstly, we 'll prove that $\sim (I_\alpha \otimes g^\beta)\Delta_{\alpha,\beta}F_{\alpha\beta} \in W_\alpha$.

$$\begin{aligned}
L_{1,\alpha}[\sim (I_\alpha \otimes g^\beta)\Delta_{\alpha,\beta}F_{\alpha\beta}] &= \sim (L_{1,\alpha} \otimes g^\beta)\Delta_{\alpha,\beta}F_{\alpha\beta} \\
&= (I_{C_1} \otimes \sim (I_\alpha \otimes g^\beta))(L_{1,\alpha} \otimes I_\beta)\Delta_{\alpha,\beta}F_{\alpha\beta} \\
&= (I_{C_1} \otimes \sim (I_\alpha \otimes g^\beta))(I_{C_1} \otimes \Delta_{\alpha,\beta})L_{1,\alpha\beta}F_{\alpha\beta} \\
&= (I_{C_1} \otimes \sim (I_\alpha \otimes g^\beta))(I_{C_1} \otimes \Delta_{\alpha,\beta})(I_{C_1} \otimes F_{\alpha\beta})\rho_{1,1} \\
&= (I_{C_1} \otimes \sim (I_\alpha \otimes g^\beta)\Delta_{\alpha,\beta}F_{\alpha\beta})\rho_{1,1}
\end{aligned}$$

Now, we will prove that Ω is coaction on W, i.e., the two diagrams

$$\begin{array}{ccc}
W_{\alpha\beta\gamma} & \xrightarrow{\Omega_{\alpha\beta,\gamma}} & W_{\alpha\beta} \otimes H_\gamma \\
\Omega_{\alpha,\beta\gamma} \downarrow & & \downarrow (\Omega_{\alpha,\beta} \otimes I_\gamma) \\
W_\alpha \otimes H_{\beta\gamma} & \xrightarrow[I_{W_\alpha} \otimes \Delta_{\beta,\gamma}]{} & W_\alpha \otimes H_\beta \otimes H_\gamma
\end{array}
\quad , \quad
\begin{array}{ccc}
W_\alpha & \xrightarrow{\Omega_{\alpha,1}} & W_\alpha \otimes H_1 \\
& \sim \searrow & \downarrow (I_{W_\alpha} \otimes \epsilon) \\
& & W_\alpha \otimes K
\end{array}$$

are commutes.

1.

$$\begin{aligned}
(\Omega_{\alpha,\beta} \otimes I_\gamma)\Omega_{\alpha\beta,\gamma}F_{\alpha\beta\gamma} &= (\Omega_{\alpha,\beta} \otimes I_\gamma)(\sim (I_{\alpha\beta} \otimes g^\gamma)\Delta_{\alpha\beta,\gamma}F_{\alpha\beta\gamma} \otimes e^\gamma) \\
&= \Omega_{\alpha,\beta}(\sim (I_{\alpha\beta} \otimes g^\gamma)\Delta_{\alpha\beta,\gamma}F_{\alpha\beta\gamma}) \otimes e^\gamma \\
&= \sim (I_\alpha \otimes g^\beta)\Delta_{\alpha,\beta} \sim (I_{\alpha\beta} \otimes g^\gamma)\Delta_{\alpha\beta,\gamma}F_{\alpha\beta\gamma} \otimes e^\beta \otimes e^\gamma \\
&= \sim (\sim \otimes I_K)((I_\alpha \otimes g^\beta)\Delta_{\alpha,\beta} \otimes g^\gamma)\Delta_{\alpha\beta,\gamma}F_{\alpha\beta\gamma} \otimes e^\beta \otimes e^\gamma \\
&= \sim (\sim \otimes I_K)(I_\alpha \otimes g^\beta \otimes g^\gamma)(\Delta_{\alpha,\beta} \otimes I_\gamma)\Delta_{\alpha\beta,\gamma}F_{\alpha\beta\gamma} \otimes e^\beta \otimes e^\gamma \\
&= \sim (\sim \otimes I_K)(I_\alpha \otimes g^\beta \otimes g^\gamma)(I_\alpha \otimes \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma}F_{\alpha\beta\gamma} \otimes e^\beta \otimes e^\gamma \\
&= \sim (\sim \otimes I_K)(I_\alpha \otimes (g^\beta \otimes g^\gamma)\Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma}F_{\alpha\beta\gamma} \otimes e^\beta \otimes e^\gamma = \xi_1
\end{aligned}$$

Also

$$\begin{aligned}(I_{W_\alpha}\otimes\Delta_{\beta,\gamma})\Omega_{\alpha,\beta\gamma}F_{\alpha\beta\gamma} &= (I_{W_\alpha}\otimes\Delta_{\beta,\gamma})(\sim(I_\alpha\otimes g^{\beta\gamma})\Delta_{\alpha,\beta\gamma}F_{\alpha\beta\gamma}\otimes e^{\beta\gamma})\\ &= \sim(I_{W_\alpha}\otimes g^{\beta\gamma})\Delta_{\alpha,\beta\gamma}F_{\alpha\beta\gamma}\otimes\Delta_{\beta,\gamma}(e^{\beta\gamma})=\xi_2.\end{aligned}$$

from Lemma 4.2. the first digram is commut

2.

$$\begin{aligned}(I_{W_\alpha}\otimes\epsilon)\Omega_{\alpha,1}F_\alpha &= (I_{W_\alpha}\otimes\epsilon)[\sim(I_\alpha\otimes g^1)\Delta_{\alpha,1}F_\alpha\otimes e^1]\\ &= \sim(I_{W_\alpha}\otimes g^1)\Delta_{\alpha,1}F_\alpha\otimes\epsilon(e^1)\\ &= \sim(I_{W_\alpha}\otimes\epsilon(e^1)g^1)\Delta_{\alpha,1}F_\alpha\otimes 1_K\\ &= \sim(I_{W_\alpha}\otimes\epsilon)\Delta_{\alpha,1}F_\alpha\otimes 1_K\\ &= F_\alpha\otimes 1_K.\end{aligned}$$

□

Now, we constract an induced and coinduced representation from subHopf $\pi-$coalgebra.

Theorem 4.4. *Let H be a finite dimentional Hopf π-coalgebra and $A=\{A_\alpha\}_{\alpha\in\pi}$ be an isolated subHopf π-coalgebra of H. If $V=\{V_\alpha\}_{\alpha\in\pi}$ is left π-comodule over A by $\rho=\{\rho_{\alpha,\beta}\}_{\alpha,\beta\in\pi}$, then we can construct an induced and coinduced representation over H.*

Proof. Since $A=\{A_\alpha\}_{\alpha\in\pi}$ is an isolated subHopf π-coalgebra of H, there exist a family $I=\{I_\alpha\}_{\alpha\in\pi}$ of Hopf π-coideal of H such that $H_\alpha=A_\alpha\oplus I_\alpha$ for all $\alpha\in\pi$. We'll prove that (A,σ) is left $\pi-$coisotropic quantum subgroup of H where $\sigma=\{\sigma_\alpha\}_{\alpha\in\pi}$ and $\sigma_\alpha: H_\alpha\to A_\alpha$ as $\sigma_\alpha(m_\alpha+i_\alpha)=m_\alpha$.

Clear, A is π-coalgebra. We 'll prove that A_α is left H_α-module for all $\alpha\in\pi$. We define $\Phi_\alpha: H_\alpha\otimes A_\alpha\to A_\alpha$ as follow $\Phi_\alpha((m_\alpha+i_\alpha)\otimes a_\alpha)=m_\alpha a_\alpha$. Then

$$\begin{aligned}\Phi_\alpha(\mu_\alpha\otimes I_\alpha)((m_\alpha+i_\alpha)\otimes(n_\alpha+j_\alpha)\otimes a_\alpha) &= \Phi_\alpha((m_\alpha n_\alpha+m_\alpha j_\alpha+i_\alpha n_\alpha+i_\alpha j_\alpha)\otimes a_\alpha)\\ &= m_\alpha n_\alpha a_\alpha\end{aligned}$$

and

$$\begin{aligned}\Phi_\alpha(I_\alpha\otimes\Phi_\alpha)((m_\alpha+i_\alpha)\otimes(n_\alpha+j_\alpha)\otimes a_\alpha) &= \Phi_\alpha((m_\alpha+i_\alpha)\otimes n_\alpha a_\alpha)\\ &= m_\alpha n_\alpha a_\alpha\end{aligned}$$

where $m_\alpha j_\alpha + i_\alpha n_\alpha + i_\alpha j_\alpha \in I_\alpha$
Also, we have

$$\begin{aligned}\Phi_\alpha(\eta_\alpha \otimes I_\alpha)(k \otimes a_\alpha) &= \Phi_\alpha((\eta_\alpha(k) + 0) \otimes a_\alpha) \\ &= \eta_\alpha(k) a_\alpha = k a_\alpha.\end{aligned}$$

Now, we will prove that σ_α is $\pi-$coalgebra map

$$\begin{aligned}(\sigma_\alpha \otimes \sigma_\beta)\Delta_{\alpha,\beta}(h_{\alpha\beta}) &= (\sigma_\alpha \otimes \sigma_\beta)\Delta_{\alpha,\beta}(m_{\alpha\beta} + i_{\alpha\beta}) \\ &= (\sigma_\alpha \otimes \sigma_\beta)(\Delta_{\alpha,\beta}(m_{\alpha\beta}) + \Delta_{\alpha,\beta}(i_{\alpha\beta})) \\ &= \Delta_{\alpha,\beta}(m_{\alpha\beta})\end{aligned}$$

and

$$\begin{aligned}\Delta_{\alpha,\beta}\sigma_{\alpha\beta}(h_{\alpha\beta}) &= \Delta_{\alpha,\beta}\sigma_{\alpha\beta}(m_{\alpha\beta} + i_{\alpha\beta}) \\ &= \Delta_{\alpha,\beta}(m_{\alpha\beta}).\end{aligned}$$

We will prove that σ_α is left module map

$$\begin{aligned}\Phi_\alpha(I_\alpha \otimes \sigma_\alpha)(h_\alpha \otimes k_\alpha) &= \Phi_\alpha(I_\alpha \otimes \sigma_\alpha)((m_\alpha + i_\alpha) \otimes (n_\alpha + j_\alpha)) \\ &= \Phi_\alpha((m_\alpha + i_\alpha) \otimes n_\alpha) = m_\alpha n_\alpha\end{aligned}$$

and

$$\sigma_\alpha\mu_\alpha(h_\alpha \otimes k_\alpha) = \sigma_\alpha(m_\alpha n_\alpha + m_\alpha j_\alpha + i_\alpha n_\alpha + i_\alpha j_\alpha) = m_\alpha n_\alpha$$

Therefore (A, σ) is left $\pi-$coisotropic quantum subgroup of H. Since V is left $\pi-$comodule over A, then we can construct an induced and coinduced representation over H. □

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